Séance 3 : calcul d'intégral

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Calculer les primitives suivantes :

a) $\int \frac{x^2}{1+x^2}\,dx$ b) $\int \frac{2x+1}{x^2(x+1)^2}\,dx$ c) $\int \sin^2 x\,dx$

d) $\int \frac{1+x+x^2}{\sqrt{1-x^2}}\,dx$ e) $\int \frac{dx}{\cos^2 x \sin^2 x}$ f) $\int \frac{dx}{\sin x}$

g) $\int \frac{dx}{5\cosh x+3\sinh x+4}$ h) $\int \frac{e^{\mathrm{Arc}\sin x}}{\sqrt{1-x^2}}\,dx$ i) $\int \frac{\left(\ln\left|\ln|x|\right|\right)^2}{x\ln|x|}\,dx$

j) $\int \frac{1-\cos\frac{x}{3}}{\sin\frac{x}{2}}\,dx$ k) $\int \frac{\sqrt{x^{10}+1}}{x}\,dx$ l) $\int \frac{\tan x}{2+\tan^2 x}\,dx$

m) $\int x^4 \ln|x|\,dx$ n) $\int (x^3+1)\,e^{-x}\,dx$ o) $\int (\mathrm{Arcsin}\,x)^2\,dx$

p) $\int \sinh x \sin x\,dx$ q) $\int \sinh^2 x \sin^2 x\,dx$ r) $\int \ln(x^2+4x+5)\,dx$

s) $\int x\sqrt{-x^2+3x-2}\,dx$ t) $\int \frac{x}{\sqrt{x^2+x+2}}\,dx$ u) $\int \frac{dx}{\sin x\sqrt{1-\cos x}}$

a) $$\int \frac{x^2}{1+x^2}\,dx = \int \frac{x^2+1-1}{x^2+1}\,dx$$

$$= \int \frac{x^2+1}{x^2+1}dx - \frac{1}{x^2+1}\,dx$$

$$= \int 1.\,dx - \int \frac{dx}{x^2+1}$$

$$= x - \mathrm{Arctg}(x) + cte.$$

b) $$\int \frac{2x+1}{x^2.(x+1)^2}\,dx$$

$$x.F(x) = \frac{2x+1}{(x^2(x+1)^2)} = x\frac{a}{x} + \frac{b}{x^2}\cancel{x} + \frac{c}{(x+1)}x + \frac{d}{(x+1)^2}x$$

$$b = x^2.F(x)\Big|_{x=0} = \frac{1}{1} = 1.$$

$$d = (x+1)^2 F(x)\Big|_{x=-1} = -1$$

c) $$\int \sin^2 x\,dx = \int \frac{1}{2}\left(1-\cos(2x)\right)dx.$$

$$= \frac{1}{2}.\left\{\int dx - \int \cos(2x)\,dx\right\}$$

$$= \frac{1}{2}.\left\{x - \frac{1}{2}.\sin(2x)\right\}.$$

$$* \int \frac{dx}{\sqrt{1-x^2}} = \operatorname{Arcsin}(x) + cte.$$

$$* \int \frac{x}{\sqrt{1-x^2}} dx = -\int \frac{(1-x^2)'}{2\sqrt{1-x^2}} dx \qquad \int \frac{f'}{2\sqrt{f}} \to \sqrt{f}.$$

$$= -\sqrt{1-x^2} + cte$$

$$* \int \frac{x^2}{\sqrt{1-x^2}} dx = -\int \frac{1-x^2-1}{\sqrt{1-x^2}} = -\int \frac{(1-x^2)}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}} dx$$

$$= -\int \sqrt{1-x^2}\, dx + \int \frac{dx}{\sqrt{1-x^2}}$$

$$\qquad\qquad\qquad\qquad \overset{\shortparallel}{\operatorname{Arcsin}(x)}$$

$\cos^2 x . \sin^2(x)$

$$* \int \sqrt{1-x^2}\, dx = \int \sqrt{1-\cos^2 t} . (-\sin(t))\, dt \qquad x = \cos t \Rightarrow dx = -\sin(t)\, dt.$$

$$t = \operatorname{Arcos}(x)$$

$$\frac{1}{\cos^2(x).\sin^2(x)} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x).\sin^2(x)} \sin t\, dt = -\int \sin^2 t\, dt$$

$$\left\| \int \frac{1 \mp x + x^2}{\sqrt{1-x^2}} \quad \frac{\cos^2(x)}{\cos^2(x)\sin^2(x)} + \frac{\sin^2(x)}{2.\cos^2(x).\sin^2(x)} - \sqrt{1-x^2}. - \frac{1}{2}.\operatorname{Arcos}(x) + \frac{1}{2} * \sqrt{1-x^2}. \right\|$$

$$= \frac{1}{\sin^2(x)} + \frac{1}{\cos^2(x)}$$

e) dx

$$\int \frac{1}{\cos^2(x).\sin^2(x)} dx = \int \frac{1}{\sin^2(x)} dx + \int \frac{1}{\cos^2(x)} dx$$

$$= -\cot g(x) + tg(x) + cte.$$

$$* \int \frac{2}{2\cos^2(x).\sin^2(x)} = \int \frac{2}{\sin^2(2x)} dx = \int \frac{d(2x)}{\sin^2(2x)} \qquad t = 2x \Rightarrow dt = 2.dx \Rightarrow dx = \frac{dt}{2}$$

$$= \int \frac{dt}{\sin^2(t)} = -\cot g(t) + cte$$

$$= -\cot g(2x) + cte.$$

* $\| t = tg\left(\frac{x}{2}\right) \|$

$$t = tg\left(\frac{x}{2}\right) \quad , \quad dt = \frac{1+t^2}{2} \cdot dx .$$

$$dx = \frac{2}{1+t^2} \cdot dt .$$

$$\sin(x) = \frac{2t}{1+t^2} \Rightarrow \frac{1}{\sin(x)} = \frac{1+t^2}{2t}$$

$$\int \frac{1}{\sin(x)} dx = \int \frac{\cancel{1+t^2}}{\cancel{2}t} \frac{\cancel{2}}{\cancel{1+t^2}} \cdot dt$$

$$= \ln|t| + cte .$$

f)

$$\int \frac{dx}{\sin(x)} = \ln\left|tg\left(\frac{x}{2}\right)\right| + cte .$$

$$t = th\left(\frac{x}{2}\right) \quad , \quad dt = \frac{1}{2} \cdot (1-t^2)\, dx .$$

$$dx = \frac{2}{1-t^2} \cdot dt .$$

$$ch(x) = \frac{1+t^2}{1-t^2} \qquad \sinh(x) = \frac{2t}{1-t^2}$$

$$\sinh(y) = \frac{e^y - e^{-y}}{2} \qquad \frac{\sinh(ix)}{i} = \frac{e^{ix} - e^{-ix}}{2i} = \sin(x) .$$

$$\sinh(ix) = i \sin(x)$$

g)

$$ch(ix) = \frac{e^{ix} - e^{-ix}}{2} = \cos(x)$$

$$\int \frac{1}{\left(5 \cdot \frac{(1+t^2)}{\cancel{1-t^2}} + 3 \frac{2t}{\cancel{1-t^2}} + 4\right)} \cdot \frac{2}{\cancel{1-t^2}} \cdot dt$$

$$= 2 \cdot \int \frac{-1}{5 + 5t^2 + 6t + 4(1-t^2)} dt$$

$$= 2 \int \frac{dt}{t^2 + 6t + 9} = 2 \cdot \int (t+3)^{-2} \cdot dt .$$

$$\frac{2}{1 \cdot ch(\frac{x}{2})}$$

$$= \frac{-2}{\frac{sh(\frac{x}{2})}{ch(\frac{x}{2})} - 3} + cte = \frac{-2\, ch(\frac{x}{2})}{sh(\frac{x}{2}) - 3} + cte.$$

h) $\int \frac{e^{Arsin(x)}}{\sqrt{1-x^2}}\,dx.$ $\quad \int f' \cdot e^{f} \to e^{f}.$

$t = Arsin(x) \Rightarrow dt = \frac{1}{\sqrt{1-x^2}} \cdot dx \Rightarrow dx = \sqrt{1-x^2} \cdot dt.$

$$\int \frac{e^t}{\cancel{\sqrt{1-x^2}}} \cancel{\sqrt{1-x^2}}\,dt = \int e^t\,dt = e^t + cte.$$

$$= e^{Arsin(x)} + cte$$

i) $\int \frac{(\ln|\ln|x||)^2}{x \cdot \ln|x|}\,dx$

$t = \ln|x| \quad dt = \frac{dx}{x}.$

$$\int \frac{(\ln|\ln|x||)^2}{\cdot\ln|x|} \cdot \frac{dx}{x} = \int \frac{(\ln|t|)^2}{t} \cdot dt.$$

$$= \int (\ln|t|)' \cdot (\ln|t|)^2\,dt. \qquad v = \ln|t| \to dv = \frac{dt}{t}$$

$$= \int v^2\,dv = \frac{v^3}{3} + cte = \frac{\ln|t|^3}{3} + cte.$$

j) $\int \frac{1-\cos(\frac{x}{3})}{\sin(\frac{x}{2})}\,dx.$

$$\sin\left(\frac{x}{2}\right) = \sin\left(\frac{x}{3} + \frac{x}{6}\right) = \sin\left(\frac{x}{3}\right) \cdot \cos\left(\frac{x}{6}\right) + \cos\left(\frac{x}{3}\right) \cdot \sin\left(\frac{x}{6}\right)$$

$$\rightarrow dx = \frac{6}{1+t^2} \cdot dt$$

$$\cos\left(\frac{x}{3}\right) = \boxed{\frac{1-t^2}{1+t^2}}, \quad 1-\cos\left(\frac{x}{3}\right) = 1 - \frac{1-t^2}{1+t^2} = \frac{2t^2}{1+t^2} = t \cdot \frac{2t}{1+t^2} = t \cdot \sin\left(\frac{x}{3}\right)$$

$$\cos\left(\frac{x}{6}\right) = \quad \cos(\theta) \quad \cos\left(\frac{\theta}{2}\right)$$

$$\cos^2\theta = \frac{1 + \cos(2\theta)}{2}$$

$$\cos\left(\frac{x}{6}\right) = \pm\sqrt{\frac{1+\cos\left(\frac{x}{3}\right)}{2}} = \pm\sqrt{\frac{1+\frac{1-t^2}{1+t^2}}{2}}$$

$$\left\| \cos\left(\frac{x}{6}\right) = \sqrt{\frac{\frac{\cancel{2}}{1+t^2}}{\cancel{2}}} = \sqrt{\frac{1}{1+t^2}} \right\|$$

$$\sin^2\left(\frac{x}{6}\right) = \frac{1-\cos\left(\frac{x}{3}\right)}{2} = \frac{1}{2} \cdot t \cdot \sin\left(\frac{x}{3}\right)$$

$$\left\| \sin^2\left(\frac{x}{6}\right) = \frac{t^2}{1+t^2} \right\|$$

k) $$\int \sqrt{x^{10}+1} \cdot \frac{dx}{x}.$$

$$t = x^5 \rightarrow \frac{dt}{t} = \frac{5 \cdot x^4}{x^5} dx = 5 \frac{dx}{x}$$

$$\frac{dx}{x} = \frac{1}{5} \cdot \frac{dt}{t}$$

$$\frac{1}{5}\int \sqrt{t^2+1} \, \frac{dt}{t} \qquad u = t^2+1 \rightarrow du = 2t\,dt \rightarrow \left| dt = \frac{du}{2t} = \frac{du}{2\sqrt{u-1}} \right|$$

$$\hookrightarrow t = \sqrt{u-1}$$

$$\frac{1}{5}\int \frac{\sqrt{u}}{\sqrt{u-1}} \, \frac{du}{2\sqrt{u-1}}$$

$$\frac{1}{10}\int \frac{\sqrt{u}}{(u-1)} \, du. \Rightarrow \frac{1}{10} \cdot \int \frac{v}{v^2-1} \cdot 2v \cdot dv = \frac{1}{5} \cdot \int \frac{v^2}{v^2-1} dv.$$

$$= \frac{1}{5} \cdot v + \int \frac{dx}{v^2-1}$$

$$\int \frac{dv}{v^2-1} = \int \frac{dx}{(v-1)(v+1)} = \int \frac{1}{2}\frac{dx}{v-1} - \frac{1}{2}\int \frac{dx}{v+1}$$

$$\frac{1+0v}{(x-1)(v+1)} = \frac{a}{v-1} + \frac{b}{v+1} = \frac{a(v+1)+b(v-1)}{(v-1)(v+1)} = \frac{v(a+b)+a-b}{(v-1)(v+1)}$$

$a+b=0 \Rightarrow a=-b$

$a-b=1$

$-2b=1$

$$\begin{cases} b=-1/2 \\ a=1/2 \end{cases}$$

e) $t=tg\left(\frac{x}{2}\right)$

$$\cos(x) = \frac{1-t^2}{1+t^2} \qquad dx =$$

$$\sin(x) = \frac{2t}{1+t^2}$$

$$tg(x) = \frac{2t/\cancel{1+t^2}}{1-t^2/\cancel{1+t^2}} = \frac{2t}{1-t^2}$$

$$\int \frac{\frac{2t}{1-t^2}}{2+\left(\frac{2t}{1-t^2}\right)^2} \cdot \frac{2}{1+t^2} \cdot dt$$

$$\int \frac{\frac{2t}{\cancel{1-t^2}}}{\frac{2(1-t^2)^2+4t^2}{(1-t^2)^{\cancel{2}}}} \frac{2}{1+t^2} \cdot dt = \int \frac{2t(1-t^2)}{\left(2(1-t^2)^2+4t^2\right)} \cdot \frac{2}{(1+t^2)} \cdot dt$$

$$\int \frac{\frac{\sin(x)}{\cancel{\cos(x)}}}{2+\frac{\sin^2(x)}{\cos^2(x)}} dx = \int \frac{\sin(x) \cdot \cos(x)}{2\cos^2(x)+\sin^2(x)} dx$$

$$\begin{cases} t = \cos(x) \Rightarrow x = \operatorname{Arccos}(t) \\ u = \cos^2(x)+1 \end{cases} \quad = -\frac{1}{2}\int \frac{(t^2+1)'}{t^2+1}\,dt \quad -\frac{1}{2}\cdot \ln|t^2+1| + ct_e$$

$$-\frac{1}{2}\ln|\cos^2(x)+1| + ct_e .$$

o) $t = \operatorname{Arcsin}(x)$

p) $\sinh(x) = \frac{e^x - e^{-x}}{2}$

$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$

$\int e^{(ai+b)x} - e^{(ci+d)x}.$

$\int e^{zx} = \frac{1}{z}\cdot e^{zx} \qquad z \in \mathbb{C}.$

q) $\sinh^2(x) = \frac{1}{2}(\cosh(2x) - 1)$

$\sin^2(x) = \frac{1}{2}\cdot(1 - \cos(2x))$

$\int \cosh(2x)\cos(2x)\,dx$

r) $\int \ln(x^2+4x+5)\,dx = \int \ln((x+2)^2+1)\,dx$

$t = x+2 \Rightarrow dt = dx \qquad \int 1\cdot\ln(t^2+1)\,dt. =$ Intégration par partie.

s) $\int x\cdot\sqrt{-x^2+3x-2} = \int x\sqrt{(x^2-3x+2)}$

$$\sqrt{x^2+x+2} = \sqrt{x^2 + 2\cdot\frac{1}{2}\cdot x + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 2} \qquad 2-\frac{1}{4} =$$

$$= \sqrt{\left(x+\frac{1}{2}\right)^2 + \frac{7}{4}} = \sqrt{\frac{7}{4}\cdot t^2 + \frac{7}{4}} = \frac{\sqrt{7}}{2}\sqrt{t^2+1}.$$

$$\sqrt{\frac{7}{4}}\, t = x + 1/2 \qquad = \frac{2x+1}{\sqrt{7}} \Rightarrow x = \frac{1}{2}\cdot(t\sqrt{7}-1) \Rightarrow dx = \frac{\sqrt{7}}{2}\cdot dt.$$

$$\frac{x}{\sqrt{x^2+x+2}}\,dx = \frac{\frac{1}{2}\cdot(t\sqrt{7}-1)}{\cancel{\frac{\sqrt{7}}{2}}\cdot\sqrt{t^2+1}}\cdot\cancel{\frac{\sqrt{7}}{2}}\cdot dt$$

$$= \frac{1}{2}\int \frac{t\sqrt{7}-1}{\sqrt{t^2+1}}\cdot dt = \frac{\sqrt{7}}{2}\cdot\int\frac{t\,dt}{\sqrt{t^2+1}} - \frac{1}{2}\cdot\int\frac{1}{\sqrt{t^2+1}}\,dt$$

$$* \int \frac{t}{\sqrt{t^2+1}}\,dt = \frac{1}{2}\int\frac{(t^2+1)'}{\sqrt{t^2+1}} = \sqrt{t^2+1}$$

$$* \int\frac{1}{\sqrt{t^2+1}}\,dt = \int du = u + cte \qquad \sqrt{1-u^2} \Rightarrow t=\cos(x),\ t=\sin(x)$$

$$\boxed{\sqrt{1+t^2},\ cte} \qquad \sqrt{1+u^2} \qquad t = sh(x)$$

on pose: $u = \sinh(t)$

$$du = \cosh(t)\,dt. \qquad \cos^2(x)+\sin^2(x) = 1$$

$$= \sqrt{1+sh^2(t)} \qquad ch^2(x) - sh^2(x) = 1$$

$$ch^2(x) = 1 + sh^2(x)$$

$\int F(x)\,dx$: décomposition

$=$

$1 - \cos(x) = 2 \sin^2(x/2)$

$\sin(x) = 2 \sin\left(\frac{x}{2}\right) \cos(x/2)$

$u = \sin\left(\frac{x}{2}\right)$

$\sqrt{ax^2+bx+c} = \sqrt{(Ax+B)^2+C}$

$t = \frac{Ax+B}{\sqrt{C}}$

$(\Delta < 0)$