Séance 10 : Correction Examen

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Examen d'analyse II , SMPC(Analyse II)- durée 1h 30min.

Exercice I : On se propose de calculer $I = \int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx$.

a) Montrer que : $I = \int_0^\pi \frac{(\pi-u)\sin u}{1+\cos^2 u}du$ poser $(x = \pi - u)$

b) En déduire que : $I = \frac{\pi}{2}\int_0^\pi \frac{\sin u}{1+\cos^2 u}du$

c) Calculer la valeur de I.

Exercice II :

1. Pour $\alpha > 0$ montrer que l'intégrale suivante converge : $\int_1^{+\infty} \frac{\sin(t)}{t^\alpha}dt$.
2. On propose de calculer la valeur des intégrales suivantes :

$$I = \int_0^{+\infty} \frac{dt}{1+t^4} \quad ; \quad J = \int_0^{+\infty} \frac{t^2}{1+t^4}dt.$$

a) Vérifier que I et J sont convergentes.

b) Montrer que :

$$\forall a > 0 \quad \int_{\frac{1}{a}}^{a} \frac{dt}{t^4+1} = \int_{\frac{1}{a}}^{a} \frac{t^2}{t^4+1}dt.$$

c) En déduire que $I = J$.

d) Montrer que : $2I = \int_0^{+\infty} \frac{t^2+1}{1+t^4}dt$.

e) En posant $t = e^s$; montrer que $2I = \int_{-\infty}^{+\infty} \frac{\operatorname{ch} s}{1+2\operatorname{sh}^2(s)}ds$.

f) en utilisant un changement de variable convenable calculer les valeurs de I et J.

Exercice III : On pose $f(t) = \frac{1}{t^4-1}$.

a) Montrer que :

$$f(t) = \frac{1}{4(t-1)} + \frac{1}{4(t+1)} - \frac{t}{2(t^2+1)}.$$

b) En déduire les primitives de la fonction f.

$$I = \int_0^\pi \frac{x\sin(x)}{1+\cos^2(x)}\cdot dx$$

a) $x = \pi - u$ $\qquad I = \int_0^\pi \frac{(\pi-u)\sin(u)}{1+\cos^2 u}du$

$dx = -du.$

$$\begin{cases} \text{si } x = \pi : & u = 0 \\ \text{si } x = 0 : & u = \pi. \end{cases}$$

$$I = \int_\pi^0 \frac{(\pi-u).\sin(\pi-u)}{1+\cos^2(\pi-u)}\cdot(-du) = \int_0^\pi \frac{(\pi-u)\sin u}{1+\cos^2(u)}du$$

b) $I = \frac{\pi}{2}\cdot\int_0^\pi \frac{\sin u}{1+\cos^2 u}du$

$$I = \pi\int \frac{\sin(u)}{1+\cos^2 u}du - \int \frac{u.\sin u}{1+\cos^2 u}\cdot du. \qquad I = \int f(x)\,d$$

$$2I = \pi\int \frac{\sin(u)}{1+\cos^2 u}du \qquad I$$

$$I = \frac{\pi}{2}\cdot\int_0^\pi \frac{\sin u}{1+\cos^2 u}\cdot du.$$

c) $I = \frac{\pi}{2}\cdot\int_0^\pi \frac{\sin u}{1+\cos^2 u}du$ $\qquad u = tg\left(\frac{t}{2}\right).$

$$\operatorname{Arctg}' x = \frac{1}{1+x^2} \qquad f(g(x))' = g'(x)\cdot f'$$

$$\operatorname{Arctg}(\cos(x))' = \frac{-\sin(x)}{1+\cos^2 x}$$

$$I = \frac{\pi}{2}\int -(\operatorname{Arctg}(\cos(x)))'.$$

$$= \frac{\pi}{2}\cdot\left[-\operatorname{Arctg}(\cos(x))\right]_0^\pi$$

$$= -\frac{\pi}{2}\cdot\left\{\operatorname{Arctg}(\cos(\pi)) - \operatorname{Arctg}(\cos(0))\right\}$$

$$= -\frac{\pi}{2}\left(\operatorname{Arctg}(-1) - \operatorname{Arctg}(1)\right)$$

$$= \frac{\pi^2}{4}.$$

Ex II : $\qquad \int_1^{+\infty} \frac{\sin(t)}{t^\alpha}dt. \qquad \alpha > 0$

$$= \int f(t)\,dt$$

$$= \int f(u)\,du$$

$$(g(x))$$

$$* \quad \int_1^a \frac{\sin t}{t^\alpha} dt = \left[- \frac{\cos(t)}{t^\alpha} \right]_1^a - \alpha \cdot \int \frac{\cos(t)}{t^{\alpha+1}} dt.$$

admet une limite. CV

*i) $\lim_{+\infty} \frac{\cos(t)}{t^\alpha} = 0.$

ii) $\left| \frac{\cos(t)}{t^{\alpha+1}} \right| \leq \frac{1}{t^{\alpha+1}}$ $\alpha+1 > 1$

$\int_1^{+\infty} \frac{\sin t}{t^\alpha}$: CV

2) $I = \int_0^{+\infty} \frac{dt}{1+t^4}$ $J = \int_0^{+\infty} \frac{t^2}{1+t^4} dt$

a) $f(t) = \frac{1}{1+t^4}$ est continue sur $[0, +\infty[$ et positive sur [

$\frac{1}{1+t^4} \underset{+\infty}{\sim} \frac{1}{t^4}$ or $\int_\varepsilon^{+\infty} \frac{dt}{t^4}$ CV $(\alpha = 4 > 1)$

alors : $\int_0^{+\infty} \frac{dt}{1+t^4}$ est CV

$* \int \frac{t^2}{1+t^4} dt$: $\frac{t^2}{1+t^4} \underset{+\infty}{\sim} \frac{t^2}{t^4} = \frac{1}{t^2}$ $(\alpha = 2$

$\int_\varepsilon^{+\infty} \frac{dt}{t^2}$ CV ce qui implique que $\int_0^{+\infty} \frac{t^2}{1+t^4} dt$

b) $\forall a > 0$: $\int_{\frac{1}{a}}^a \frac{dt}{t^4+1} = \int_{1/a}^a \frac{t^2}{t^4+1} dt$

$t = \frac{1}{x}$ $dt = -\frac{dx}{x^2}$

$t = a$ $x = \frac{1}{a}$ $\int_a^{1/a} \frac{-\frac{dx}{x^2}}{\frac{1}{x^4}+1}$

$t = \frac{1}{a}$ $x = a.$

$$= \int_{\frac{1}{a}}^a \frac{x^2 dx}{1+x^4} \qquad \int f(x) d$$

$t = x$

$$= \int_{1/a}^a \frac{t^2 . dt}{1+t^4}. \quad (\text{cqfd}).$$

c) $I = J$:

$0, +\infty[$

$> 1)$

cv

$n = \int f(t)\,dt$

c) $I = J$:

$$\int_{1/a}^{a} \frac{dt}{1+t^4} = \int_{1/a}^{a} \frac{t^2 \cdot dt}{1+t^4}.$$

Si $a \to +\infty$ $$\int_0^{+\infty} \frac{dt}{1+t^4} = \int_0^{+\infty} \frac{t^2\,dt}{1+t^4}$$

$$\boxed{I = J.}$$

d) $$I + J = 2I = \int_0^{+\infty} \frac{dt}{1+t^4} + \int_0^{+\infty} \frac{t^2\,dt}{1+t^4}$$

$$\| \quad 2I = \int_0^{+\infty} \frac{(1+t^2)\,dt}{1+t^4}. \quad \|$$

$\Rightarrow$

e) $t = e^x$; $$2I = \int_{-\infty}^{+\infty} \frac{1+e^{2x}}{1+e^{4x}} e^x \cdot dx$$

$dt = e^x \cdot dx$

$$\begin{cases} t = 0 \; ; & x \to -\infty \\ t = +\infty & x \to +\infty \end{cases}$$

$$ch(x) = \frac{e^x + e^{-x}}{2}$$

$$sh(x) = \frac{e^x - e^{-x}}{2}$$

$$\frac{1+e^{2x}}{1+e^{4x}} e^x = \frac{e^{-x}}{e^{-x}} \cdot \frac{1+e^{2x}}{1+e^{4x}} \cdot e^x$$

$$= \frac{e^{-x} + e^x}{e^{-x} + e^{3x}} \cdot \underline{\underline{e^x}}$$

$$= \frac{e^{-x} + e^x}{e^{-2x} + e^{2x}}$$

$$= \frac{\cancel{2} \cdot ch(x)}{\cancel{2} \cdot sh(2x)} = \frac{ch(x)}{ch(2x)}$$

$$\| ch(2x) = 1 + 2 \cdot sh^2(x) \| .$$

$$I = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{ch(x)}{1 + 2 \cdot sh^2(x)} dx = \int \frac{d(sh(x))}{1 + 2 sh^2 \ldots}$$

$t = sh(x)$

$$I = \frac{1}{2} \cdot \int_{-\infty}^{+\infty} \frac{dt}{1+2t^2}$$

$$\int_{-a}^{+a} \frac{dt}{1+2t^2} = \frac{1}{\sqrt{2}} \operatorname{Arctg}(\sqrt{2}t)$$

$$\lim_{a \to +\infty} \left(\frac{1}{\sqrt{2}} \cdot \operatorname{Arctg}(\sqrt{2}t) \right) = \frac{\pi}{\sqrt{2}}$$

$$\lim_{x \to +\infty} \operatorname{Arctg}(x) = \frac{\pi}{2}$$

$$\lim_{x \to -\infty} \operatorname{Arctg}(x) = -\frac{\pi}{2} .$$

Ex3

$$f(t) = \frac{t}{t^4 - 1}$$

$$t^4 - 1 = (t^2 - 1)(t^2 + 1) = (t-1)(t+1)(t^2+1) \checkmark$$

$(\Delta < 0)$

$$f(t) = \frac{t}{(t-1)(t+1)(t^2+1)} = \frac{a}{t-1} + \frac{b}{t+1} + \frac{ct}{t^2+1} \cdot t$$

$$\left\{ \begin{array}{l} (t-1) f(t) \Big|_{t=1} = \frac{1}{4} = a \\ (t+1) f(t) \Big|_{t=-1} = \frac{1}{4} = b \\ f(0) = 0 = \cancel{\frac{a}{-1}} + \cancel{\frac{b}{1}} + \frac{d}{1} \Rightarrow \boxed{d = 0} \checkmark \end{array} \right.$$

$$\lim_{+\infty} t f(t) = \frac{t^2}{t^4} = \frac{1}{t^2} = 0 = a + b + c .$$

$$c = -a - b = -\frac{1}{2}$$

10) $\int f(t)\, dt = \frac{1}{\ldots} \int \frac{dt}{\ldots} + \frac{1}{\ldots} \int \frac{dt}{\ldots} - \frac{1}{2} \cdot \int \ldots$

(x)

$\frac{(\ldots^2+1)'}{\ldots}$

b°)

$$\int f(t)\,dt = \frac{1}{4}\int \frac{dt}{t-1} + \frac{1}{4}\int \frac{dt}{t+1} - \frac{1}{2}\cdot\int \frac{(t}{}$$

$$= \frac{1}{4}\cdot\Big(\ln|t-1| + \ln|t+1| - \ln(t^2+1)$$

$$\frac{(t^2+1)'}{t^2+1}$$

$$) + C$$