Intégrales abéliennes.

$$I = \int \frac{dx}{\sqrt{ax^2+bx+c}}$$

* $a = 0$: , $t = bx+c$

$$I = \frac{1}{b}\int \frac{dt}{\sqrt{t}} = \frac{2}{b}\sqrt{bx+c}$$

* $a \neq 0$:

on écrit le trinôme sous la forme canonique :

$$ax^2+bx+c = a\left[\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}\right]$$

3 cas à distinguer :

$$\int \frac{du}{\sqrt{k^2-u^2}} = \text{Arcsin}\left(\frac{u}{k}\right)$$

$$\int \frac{du}{\sqrt{u^2+k^2}} = \text{Argsh}\left(\frac{u}{k}\right)$$

$$\int \frac{du}{\sqrt{u^2-k^2}} = \ln\left|u+\sqrt{u^2-k^2}\right| . \text{ valable si } u > k .$$

* $$I = \int \frac{dx}{(px+q)\sqrt{ax^2+bx+c}}$$

$t = \frac{1}{px+q}$: on se ramène au cas précédent.

* si $t > 0$: $\frac{dx}{px+q} = -\frac{dt}{t}$ $\quad x = \frac{1-qt}{pt}$

et $\sqrt{ax^2+bx+c} = \frac{1}{pt}\sqrt{a(1-qt)^2+b(1-qt)pt+cp^2t^2}$

Finalement : $$I = -\int \frac{p}{\sqrt{(aq^2-bqp+cp^2)t^2+(bp-2aq)t+a}}\,dt$$

(cette expression est simple si on a les valeurs des paramètres)

Exp. $$I = \int \frac{dx}{x(1+x^2)^{1/2}} \quad x > 0$$

$$I = \int \sqrt{ax^2+bx+c}\;dx$$

* $a = 0$: $t = bx+c$

$$I = \int \frac{1}{b}\sqrt{t}\,dt = \frac{2}{3b}.t^{3/2}$$

* $a \neq 0$: $ax^2+bx+c = a\left[\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}\right]$

on trouve 3 cas : $I = \int\sqrt{k^2-u^2}\,du$, $J = \int\sqrt{u^2+k^2}\,du$, $K = \int\sqrt{u^2-k^2}\,du$

⇒ I : on pose $u = k.\sin t$

$$I = k^2\int\sqrt{1-\sin^2 t}.\cos t\,dt$$
$$= k^2\int \cos^2 t.\,dt = k^2\left(\frac{\sin 2t}{4}+\frac{t}{2}\right).$$

⇒ J : on pose : $u = k\,\text{sh}(t)$

$$J = k^2\left(\frac{\text{sh}(2t)}{4}+\frac{t}{2}\right).$$

⇒ K : on pose : $u = k\,\text{ch}(t)$ $\quad t \geq 0$

$\Rightarrow K$: on pose : $u = k\, ch(t)$ $\quad t \geq 0$

$$K = k^2 \int sh^2(t)\, dt = \frac{k^2}{2} \int (ch(2t) - 1)\, dt$$

$$= k^2 \left(\frac{sh(2t)}{4} - \frac{t}{2} \right)$$

* $I = \int R\left(x, \sqrt{\frac{ax+b}{cx+d}}\right) dx$ où $R(x, \sqrt{ax+b})$.

on pose : $u = \sqrt{\frac{ax+b}{cx+d}}$

$$x = \frac{b - du^2}{cu^2 - a} \quad \text{et} \quad dx = 2\, \frac{ad - bc}{(cu^2 - a)^2}\, u \cdot du$$

on se ramène au calcule de fractions rationnelles.

Exp : $I = \int \frac{x^2\, dx}{(2x+1)^{1/2}}$

$$I = \int \sqrt{\frac{x-1}{x+1}}\, dx$$

$\int R\left(x, \sqrt{ax^2+bx+c}\right) dx$, R : fraction rationnelle

* $a = 0$: cas précédent.

* $a \neq 0$: ax^2+bx+c forme canonique.

$$ax^2 + bx + c = a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2} \right].$$

Posons : $k = \frac{b^2 - 4ac}{4a^2}$ et $t = x + \frac{b}{2a}$

Nous nous ramenons à la forme : $J = \int S\left(t, \sqrt{a(t^2 - k)}\right) dt$

3 Cas : i) $a > 0$, $k < 0$ on pose : $t = \sqrt{-k}\ sh(u)$

ii) $a > 0$, $k > 0$: " : $t = \sqrt{k}\ ch(u)$

iii) $a < 0$, $k < 0$: " $t = \sqrt{k} \cdot \sin(u)$.

* $k = 0$ et $a = 0$ sont triviaux.

Exemples : $I = \int \frac{dx}{1 - x^2 + 2(1 - x^2)^{1/2}}$

3.1 $\int \frac{dx}{(x-1)^{1/2}}$ 3.2 $\int \frac{dx}{(2+x)(1+x)^{1/2}}$

3.3 $\int \frac{dx}{(x^2+16x+36)^{1/2}}$ 3.4 $\int \frac{dx}{x^{1/2}(1+x)^{1/2}}$

3.5 $\int (3x^2+5)^{1/2}\, dx$ 3.6 $\int \frac{(1-x)\, dx}{(1-x^2)^{1/2}}$

3.7 $\int \frac{dx}{x+(x-1)^{1/2}}$ 3.8 $\int x(a^2+x^2)^{1/2}\, dx$

3.9 $\int \frac{x+1}{(x^2+2x)^{1/2}}\, dx$ 3.10 $\int \frac{dx}{(1+x^2)(1+x^2)^{1/2}}$

3.11 $\int \frac{dx}{(1-x)\, x^{1/2}}$ 3.12 $\int \frac{dx}{(1-x^2)(1-x^2)^{1/2}}$

3.13 $\int \frac{dx}{(-x^2-12x+8)^{1/2}}$ 3.14 $\int \frac{dx}{x^2(a^2+x^2)^{1/2}}$

3.15 $\int \frac{dx}{x+x^{1/3}}$ 3.16 $\int \frac{x^3\, dx}{(x^3+2)^{1/6}}$

3.17 $\int \frac{(x+3)\, dx}{(x^2+6x)^{1/3}}$ 3.18 $\int \frac{4x-5}{(x^2-18x+106)^{1/2}}\, dx$

3.19 $\int x(3x+5)^{1/2}\, dx$ 3.20 $\int x^2(2x^3+9)^{1/2}\, dx$

3.21 $\int \frac{dx}{(x-1)(x^2+2x-3)^{1/2}}$ 3.22 $\int \frac{dx}{(x^2+4x)^{1/2}}$

3.23 $\int \frac{dx}{x(1-x^4)^{1/2}}$ 3.24 $\int \frac{x^2\, dx}{(1-x^6)^{1/2}}$

3.25 $\int \frac{(x^2-1)^{1/2}}{x^4}\, dx$ 3.26 $\int \frac{2x^2-7x+10}{(x^2-6x+6)^{1/2}}\, dx$

$u' = ch(x) \rightarrow u = sh(x)$

$$I = \int \underset{u'}{sh(x)} \, \underset{v}{\sin(x)} \, dx .$$

$u' = sh(x) \rightarrow u = ch(x)$

$v = \sin(x) \rightarrow v' = \cos(x)$

$$I = [\underset{u}{ch(x)} \cdot \underset{v}{\sin(x)}] - \int \underset{u'}{ch(x)} \, \underset{v}{\cos(x)} \, dx .$$

$$I = ch(x)\sin(x) - \left([sh(x) \cdot \cos(x)] + \underbrace{\int sh(x) \sin x \, dx}_{I} \right)$$

$$2I = ch(x)\sin(x) - sh(x)\cos(x)$$

$$I = \frac{1}{2}\left(\text{-----} \right)$$

g(x)

ln
Arctg
Arcsin
Arccos
Arg ch
Arg sh
Arg th

ALPESC

ALP → g(x), f'(x) ; ESC → g'(x), f(x)

g'(x)

exp
cos (x)
sin(x)
ch(x)
sh(x)

Exp: $\int \underset{f'}{x^2} \, \underset{g(x)}{\text{arctg}(x)} \, dx$

$f' = x^2 \rightarrow f = \frac{x^3}{3}$

$g(x) = \text{Arctg}(x) \rightarrow g'(x) = \frac{1}{1+x^2}$

$$\int \frac{\sqrt{x^{10}+1}}{x} \, dx$$

$$t = x^{10} \rightarrow dt = 10x^9 \cdot dx$$

$$\frac{dx}{x} = \frac{1}{10} \cdot \frac{dt}{x^{10}} = \frac{1}{10} \cdot \frac{dt}{t}$$

$$\frac{1}{10}\int \frac{\sqrt{t+1}}{t} \, dt$$

$$U = t+1 \rightarrow dU = dt .$$

$$\frac{1}{10}\int \frac{\sqrt{U}}{U-1} \, dU .$$

$$V = \sqrt{U} \rightarrow V^2 = U . \rightarrow dv = \frac{dU}{2\sqrt{U}} \Rightarrow dU = 2\sqrt{U} . dv$$

$$= 2v . dv .$$

$$\frac{1}{10}\int \frac{v}{v^2-1} \, 2v . dv$$

$$\frac{1}{10}\int \frac{v}{v^2-1}\, 2v.dv \qquad 2\sqrt{u} \qquad = 2v.dv.$$

$$\frac{2}{10}\int \frac{v^2}{v^2-1}\,dv = \frac{2}{10}\cdot\int \frac{v^2-1+1}{v^2-1}\,dv$$

$$= \frac{2}{10}\int \left(1+\frac{1}{v^2-1}\right)dv$$

$$= \frac{2}{10}\left(v + \int \frac{dv}{v^2-1}\right)$$

$$= \frac{2}{10}\left(v - \frac{1}{2}\ln\left|\frac{v+1}{v-1}\right|\right)$$

$$= \frac{2}{10}\left(\sqrt{u} - \frac{1}{2}\ln\left|\frac{\sqrt{u}+1}{\sqrt{u}-1}\right|\right)$$

$$= \frac{2}{10}\left(\sqrt{t+1} - \frac{1}{2}\ln\left|\frac{\sqrt{t+1}+1}{\sqrt{t+1}-1}\right|\right)$$

$$= \frac{2}{10}\left(\sqrt{x^{10}+1} - \frac{1}{2}\ln\left|\frac{\sqrt{x^{10}+1}+1}{\sqrt{x^{10}+1}-1}\right|\right)$$