Déterminer la solution générale des équations différentielles suivantes, en choisisssant pour chacune d'elles un intervalle adapté au calcul :

1) $x y' + 2 y = x^2 - 3$ 2) $(1 + x^2) y' + x y = 2 x^2 + 1$

3) $y' + y \tan x = \frac{1}{\cos x}$ 4) $x y' - x y = e^x$

5) $x y' - 2 y = \ln x$ 6) $y' + \frac{6}{x+2} y = \frac{1}{(x+2)^2}$

7) $x (x^2 + 1) y' - (x^2 - 1) y = - 2 x$ 8) $x (x - 1) y' - 2 y = x - 1$

9) $x y' - (x + 1) y = - (x^2 + 1) e^x$ 10) $y' + \frac{y}{\sqrt{1+x^2}} = 1$

solution :

1°) $xy' + 2y = x^2 - 3$.

a) SG ESSM :

$xy' + 2y = 0$

$xy' = -2y$.

$\frac{y'}{y} = -\frac{2}{x}$

$\int \frac{dy}{y} = -\int 2 \frac{dx}{x} + k$.

$\ln|y| = -2 \ln|x| + k \Rightarrow y = e^{\left(\ln \frac{1}{|x^2|} + k\right)} = \frac{1}{x^2} \cdot e^{k} = \frac{k'}{x^2}$ $\boxed{k' = e^{k}}$

$\| y = \frac{k'}{x^2} \|$, $x \neq 0$.

b) EASM :

$xy' + 2y = x^2 - 3$. (I)

(i) $y = ax^2 + bx + c$

(ii) $y' = 2ax + b$

$x \cdot (2ax + b) + 2 \cdot (ax^2 + bx + c) = x^2 - 3$

$2ax^2 + \underline{bx} + 2ax^2 + \underline{2bx} + \underline{2c} = x^2 - 3$

$4ax^2 + 3bx + 2c = x^2 - 3$

$$\begin{cases} 4a = 1 \\ 3b = 0 \\ 2c = -3 \end{cases} \Leftrightarrow \begin{cases} a = \frac{1}{4} \\ b = 0 \\ c = -\frac{3}{2} \end{cases}$$

$$y_{ASSM} = \frac{1}{4}x^2 - \frac{3}{2}.$$

c) $$y_G = y_{SGSSM} + y_{ASSM} = \frac{k}{x^2} + \frac{x^2}{4} - \frac{3}{2} \quad (k \in \mathbb{R})$$

20) $$(1+x^2)y' + xy = 2x^2+1$$

a) SGESSM:

$$(1+x^2)y' + xy = 0.$$

$$(1+x^2)\frac{dy}{dx} = -xy$$

$$\frac{dy}{y} = -\frac{x\,dx}{(1+x^2)}$$

$$\ln|y| = -\int \frac{x}{1+x^2}\,dx + k$$

$$= -\frac{1}{2}\int \frac{(x^2+1)'}{x^2+1} + k.$$

$$\ln|y| = -\frac{1}{2}\cdot\ln(x^2+1) + k. = \ln\left|(x^2+1)^{-1/2}\right| + k.$$

$$\left\| y_{SGESSM} = \frac{1}{\sqrt{x^2+1}} + k. \right\| : \quad (k \in \mathbb{R})$$

b) SG EASM:

$$y = ax^2 + bx + c$$

$$y = x \quad (a=0,\ b=1,\ c=0)$$

3) $$y' + y \cdot tg(x) = \frac{1}{\cos(x)}$$

a) SSM:

$$y' + y \cdot tg(x) = 0.$$

$$\frac{dy}{dx} = -y \cdot tg(x)$$

$$\int \frac{dy}{y} = -\int tg(x)\,dx + k.$$

$$\ln|y| = \int \frac{d(\cos(x))}{\cos(x)} + k.$$

$$\ln|y| = \ln|\cos(x)| + k' \Rightarrow y = e^{\ln|\cos(x)| + k} = e^{k'} \cdot e^{\ln|\cos x|} = \cos(x) \cdot \underbrace{e^{k'}}_{k}$$

$$\| \; y = k \cdot \cos(x) \; \| \qquad k \in \mathbb{R}.$$

b) Sol. ASM:

$$y' + y \cdot \frac{\sin(x)}{\cos(x)} = \frac{1}{\cos(x)}$$

$$\cos(x)\, y' + \sin(x)\, y = 1 \qquad \checkmark$$

$$\| \; y = \sin(x) \; \|.$$

$$y_p = k(x) \cos(x)$$

$$y_p' = k'(x) \cdot \cos(x) - k(x) \cdot \sin(x)$$

$$\cos(x)\, y_p' + \sin(x)\, y_p = 1$$

$$\cos(x)\left(k'(x)\cos(x) - \cancel{k(x) \cdot \sin(x)}\right) + \sin(x)\, \cancel{k(x)} \cos(x) = 1$$

$$k' \cdot \cos^2(x) = 1$$

$$k'(x) = \frac{1}{\cos^2(x)} = \left(tg(x)\right)'$$

$$\| \; k(x) = tg(x) \; \|$$

$$\| \; y_p = k(x) \cos(x) = \sin(x). \; \|$$

$$\| \; y_p = k(x)\cos(x) = \sin(x) \; \|$$

c) $y = \sin(x) + k \cdot \cos(x) \; : \; k \in \mathbb{R}.$

4°) $y' - y = 0 \qquad \left(y' - y = \dfrac{e^x}{x} \right)$

$$y' = +y.$$

$$\frac{dy}{dx} = y$$

$$\int \frac{dy}{y} = \int dx + k'$$

$$\ln|y| = x + k'$$

$$y = e^{x+k'} = k \cdot e^x \qquad (k \in \mathbb{R}.)$$

b) Variation de la constante:

$$y_p = k(x) \cdot e^x$$

$$y_p' = k'(x)e^x + k(x)e^x$$

$$y_p' - y_p = \frac{e^x}{x}$$

$$k' \cdot \cancel{e^x} + \cancel{k \cdot e^x} - \cancel{k e^x} = \frac{\cancel{e^x}}{x}$$

$$k' = \frac{1}{x} \Rightarrow k = \ln(x).$$

$$y_p = \ln(x) \cdot e^x$$

c) $\underline{\underline{SG}}$:

$$\| \; y = \ln(x)e^x + k \cdot e^x \; ; \; k \in \mathbb{R}$$

5) $x y' - 2y = \ln(x). \qquad y_p = a\ln(x) + b.$

ESSM : $xy' - 2y = 0$

$$y' = + \frac{2y}{x}.$$

$$\int \frac{dy}{y} = \int \frac{2}{x} \cdot dx + cte$$

$$\ln|y| = 2 \cdot \ln|x| + cte.$$

$$\| y = k.x^2 \|$$

EASM : $y_p = k(x) \cdot x^2$.

$$y'_p = k'(x) \cdot x^2 + k(x)\, 2x.$$

$$x y'_p - 2y_p = \ln(x).$$

$$x^3 k'(x) + \cancel{k(x)\, 2x^2} - \cancel{2 \cdot k(x) x^2} = \ln(x)$$

$$k'(x) = \frac{\ln(x)}{x^3}$$

$$k(x) = \int \frac{\ln(x)}{x^3} dx.$$

$$y_p = x^2 \cdot \int \frac{\ln(x)}{x^3} dx.$$

$$y_G = x^2 \cdot \int \frac{\ln(x)}{x^3} \cdot dx + kx^2. \qquad k \in \mathbb{R}.$$

$$y_p = a \ln(x) + b.$$

$$y'_p = \frac{a}{x}$$

$$x \cdot \frac{a}{x} - 2 \cdot (a \ln(x) + b) = \ln(x).$$

$$a - 2 \cdot a \ln(x) - 2b = \ln(x)$$

$$a - 2a\ln(x) - 2b = \ln(x)$$

$$\begin{cases} a - 2b = 0 \\ -2a = 1 \end{cases} \Leftrightarrow \begin{cases} b = \frac{a}{2} = -\frac{1}{4} \\ a = -\frac{1}{2} \end{cases}$$

$$\| y_p = -\frac{1}{2}\ln(x) - \frac{1}{4} = -\frac{1}{4}\left(2\ln(x) + 1\right) \|$$

$$k(x) = \int \frac{\ln(x)}{x^3}\,dx \qquad \left\| \begin{array}{l} u = \ln(x) \rightarrow du = \frac{dx}{x} \\ dv = \frac{dx}{x^3} \rightarrow v = -\frac{1}{2x^2} \end{array} \right.$$

$$k(x) = \int \frac{\ln x}{x^3}\,dx = -\frac{\ln(x)}{2x^2} + \frac{1}{2}\cdot\int \frac{dx}{x^3}$$

$$k(x) = -\frac{\ln(x)}{2x^2} - \frac{1}{4x^2} + cte$$

$$\| y_p = k \cdot x^2 = -\frac{1}{4}\left(2\ln(x) + \frac{1}{2}\right) \|$$

6) $$y' + \frac{6}{x+2}y = \frac{1}{(x+2)^2}$$

a) <u>SGSSM</u> :

$$y' + \frac{6}{x+2}y = 0$$

$$\int \frac{dy}{y} = -\int \frac{6}{x+2}\cdot dx + cte$$

$$\ln|y| = \ln|x+2|^{-6} + cte$$

$$\| y = \frac{k}{(x+2)^6} \|$$

b) <u>SP</u> :

$$y_p = \frac{k(x)}{(x+2)^6}$$

$$y'_p = \frac{k'(x)}{(x+2)^6} - 6\frac{k(x)}{(x+2)^7}$$

$$y'_p = \frac{k'(x)}{(x+2)^6} - 6\,\frac{k(x)}{(x+2)^7}$$

$$\left(\frac{k'}{(x+2)^6} - 6\cdot\cancel{\frac{k}{(x+2)^7}}\right) + 6\cdot\cancel{\frac{k(x)}{(x+2)^7}} = \frac{1}{(x+2)^2}$$

$$\frac{k'}{(x+2)^6} = \frac{1}{(x+2)^2}$$

$$k' = (x+2)^4$$

$$k(x) = \frac{1}{5}\cdot(x+2)^5$$

$$y_p = \frac{1}{5}(x+2)^{\cancel{5}}\cdot\frac{1}{(x+2)^{\cancel{6}}}$$

$$y_p = \frac{1}{5}\cdot\frac{1}{(x+2)}$$

$$y_G = \frac{1}{5}(x+2) + \frac{k}{(x+2)^6} \qquad (k \in \mathbb{R})$$