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Fiche 1 : Les intégrales de Riemann.

i) $$\lim_{n\to+\infty} \frac{b-a}{n}\sum_{1}^{n} g(x_k) = \int_a^b g(x)\,dx.$$

$$x_k = a + \frac{b-a}{n}\cdot k.$$

ii) $$\lim_{n\to+\infty} \frac{1}{n}\sum_{1}^{n} g(x_k) = \int_0^1 g(x)\,dx$$

$$x_k = \frac{k}{n}.$$

$x_1 = a + \frac{b-a}{n}$

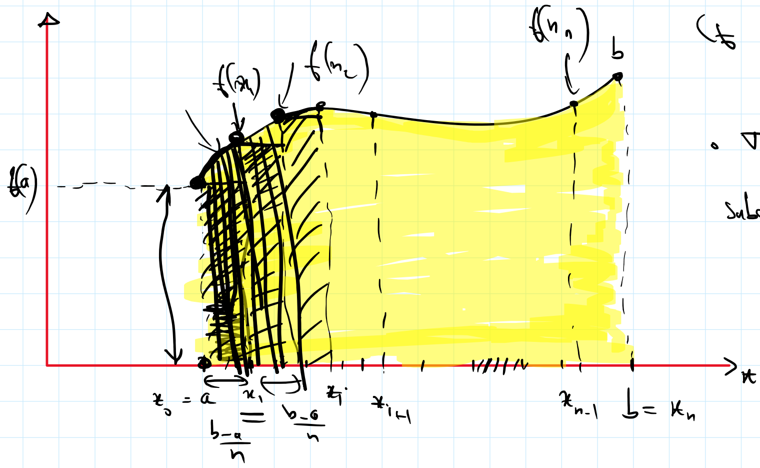
$x_2 = a + \frac{b-a}{n}\cdot 2$

$\vdots$

$x_k = a + \frac{b-a}{n}\,k.$

$f(a) \times \frac{b-a}{n}$

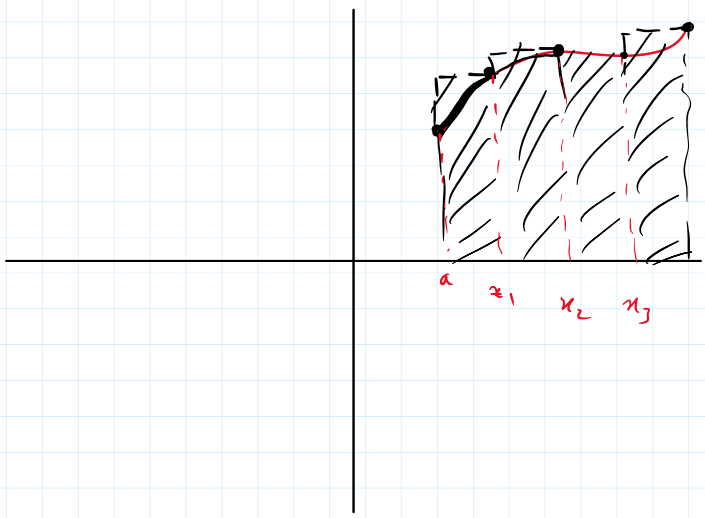
$f\left(a + \frac{b-a}{n}\right) \times \underbrace{\frac{b-a}{n}}_{\text{largeur}}$



$\sigma = \{x_0 = a, x_1, \dots, x_i, \dots, x_n = b\}$

subdivision

$\frac{b-a}{n}$



① suite arithmétique :

$$\sum^{n} U_i = U_p + U_{p+1} + \dots + U_n = \overline{U_p + U_n} \times (n-p+1)$$

$$\sum_{k=p}^{n} U_k = U_p + U_{p+1} + \dots + U_n = \frac{U_p + U_n}{2} \times (n-p+1)$$

= moyenne des termes extrêmes

× nombre de termes.

② Suite géométrique :

$$\sum_{k=p}^{n} U_p = U_p + U_{p+1} + \dots + U_n = U_p \cdot \times \frac{1-q^{n-p+1}}{1-q}$$

③ opérations sur les sommes :

$$\sum_{k=m}^{n} a = a \sum_{k=m}^{n} 1 = a \times (n-m+1) = a \times \text{nombre de termes}$$

$$\sum_{0}^{n} k = \frac{n(n+1)}{2} \qquad \sum_{0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{0}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$\sum_{0}^{n} a x_i = a(x_0 + x_1 + \dots + x_n)$$

$$\sum_{i=0}^{n} a_i = \sum_{i=0}^{p} a_i + \sum_{p+1}^{n} a_i \quad \text{(Chasles)}$$

$$\sum_{i=1}^{n} a_i = \sum_{j=0}^{n-1} a_{j+1} \quad \text{(Changement de variable)}$$

$$\sum_{i=1}^{n} a_i b_i \neq \sum_i a_i \sum_i b_i$$

$$\sum_i a_i + b_i = \sum_i a_i + \sum_i b_i$$

Calcul d'intégrals :

- $$\left|\int_a^b f(x)\, g(x)\, dx\right| \leq \left(\int_a^b |f(x)|^2\, dx\right)^{1/2} \left(\int_a^b |g(x)|^2\, dx\right)^{1/2}$$

- $$f \leq g \Rightarrow \int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx$$

- $$\left|\int_a^b f(x)\, dx\right| \leq \int_a^b |f(x)|\, dx$$

. 1er théorème de la moyenne :

$$\begin{cases} g \geq 0 \\ \text{si } m \leq f(x) \leq M \end{cases} \quad \text{alors : } m\int_a^b g(x)\,dx \leq \int_a^b f(x)\, g(x)\, dx \leq M\int_a^b g(x)\, dx$$

TVI : $\exists\ \varepsilon \in [a,b]$ tq.

$$\int_a^b f(x)\, g(x)\, dx = f(\varepsilon) \int_a^b g(x)\, dx.$$

Astuces . Calcul de intégrals .

1) Intégration par décomposition.

$$F(x) = \int (f_1(x) + f_2(x) + \dots + \dots)\, dx$$

$$= \int f_1\, dx + \int f_2(x)\, dx \dots$$

Exemple :

$$I = \int \frac{(1-\sqrt{x})^2}{\sqrt[3]{x}}\, dx$$

$$\int \frac{1 - 2\sqrt{x} + x}{\ } \quad dx$$

$$= \int \frac{1 - 2\sqrt{x} + x}{\sqrt[3]{x}}\, dx$$

$$= \int \left(\frac{1}{\sqrt[3]{x}} - 2\frac{\sqrt{x}}{\sqrt[3]{x}} + \frac{x}{\sqrt[3]{x}} \right) dx$$

$$= \int \frac{dx}{\sqrt[3]{x}} - 2 \int x^{1/2 - 1/3}\, dx + \int x^{1 - 1/3}\, dx$$

$$J = \int \frac{x - 3}{x + 1}\, dx = \int \frac{x + 1 - 4}{x + 1}\, dx$$

$$= \int \left(1 - \frac{4}{x+1} \right) dx = \int dx - 4 \int \frac{dx}{x+1}$$

$$= x - 4 \ln|x+1| + C$$

- Integration par changement de Variable :

$$I = \int \sqrt{1 - x^2}\, dx$$

$x = \sin(t) \rightarrow t = \operatorname{Arcsin}(x)$ $\qquad dx = \cos t\, dt$

$x \in [-1, 1]$ $\qquad t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$$\sqrt{1 - x^2} = \sqrt{1 - \sin^2(t)} = \sqrt{\cos^2(t)} = \cos(t)$$

$$I = \int \cos^2(t)\, dt = \int \frac{1 + \cos(2t)}{2}\, dt$$

$$= \frac{1}{2} t + \frac{\sin(2t)}{4} + C$$

$$= \frac{1}{2} \operatorname{Arcsin}(x) + \frac{\sin(2 \cdot \operatorname{Arcsin}(x))}{4} + C$$

$$J = \int \frac{dx}{\sqrt{1-x^2}}.$$

$$x = \sin(t) \Leftrightarrow t = \operatorname{Arcsin}(x)$$

$$x \in]-1,1[\quad \text{ou} \quad t \in]-\frac{\pi}{2}, \frac{\pi}{2}[$$

$$dx = \cos(t)\, dt.$$

$$J = \int \frac{\cancel{\cos(t)}\, dt}{\cancel{\cos(t)}} = \int dt = \underline{t} + C$$

$$J = \operatorname{Arcsin}(x) + C$$

Integration d'une fonction rationnelle :

Décomposition en éléments simple d'1 F.R. sur $\mathbb{R}$.

$$f(x) = \frac{P(x)}{\varphi(x)} = E(x) + \sum_{i=1}^{n} \sum_{\lambda=1}^{r} \frac{A_{i,\lambda}}{(x-a_i)^{\lambda}}$$

$$+ \sum_{k=1}^{m} \sum_{\mu=1}^{\Delta k} \frac{B_{k,\mu} x + C_{\mu}^{k}}{\left((x-\alpha_k)^2 + \beta_k^2\right)^{\mu}}$$

$E(x)$ existe si $\deg(P(x)) \geq \deg(\varphi(x))$

$$f_r(x) = \frac{x^4}{x^2 - 2x - 8}$$

i) $\varphi(x)$ factorisé dans $\mathbb{R}$. $\varphi(x) = (x+2)(x-4)$

$$\underline{f_r(x)} = \frac{x^4 \cancel{(x-4)}}{\underline{(x+2)}\,(x\cancel{-4})} = \left(E(x) + \frac{A}{x+2} + \frac{B}{x-4}\right)_{x=4}$$

$$= x^2 + 2x + 12 + \frac{A}{x+2} + \frac{B}{x-4}$$

$$A = (x+2) f_7(x) \Big|_{x=-2} = \frac{x^4}{x-4} \Big|_{x=-2} = \frac{16}{-6} = -\frac{8}{3}$$

$$B = (x-4) f_7(x) \Big|_{x=4} = \frac{x^4}{x+2} \Big|_{x=4} = \frac{4^4}{6} = \frac{4^4}{6} = \frac{128}{3}$$

$$f_7(x) = \frac{x^2+2x-3}{x^2-3x+2} = \frac{\cancel{(x-1)}(x+3)}{\cancel{(x-1)}(x-2)}$$

$$= \frac{x+3}{x-2} = \frac{x-2+5}{x-2} = 1 + \frac{5}{x-2}$$

$$f_8(x) = \frac{1}{x(x^2+x+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+x+1} + \frac{Dx+E}{(x^2+x+1)^2}$$

$$\varphi(x) = x(x^2+x+1)^2$$

$$A = x f_8(x) \Big|_{x=0}$$

$$f_8(1) = \frac{1}{3^2} = A + \frac{B+C}{3} + \boxed{\frac{D+E}{9}}$$

$$f(x) = \frac{P(x)}{\varphi(x)} = \int E(x) + \int \frac{A}{(x-a)^\lambda} + \int \sum \frac{Bx+C}{(x^2+x+1)^\mu}$$

$$\int \frac{A\,dx}{(x-a)} = A \ln|x-a| + C$$

$$\int \frac{A\,dx}{(x-a)^n} \quad n=1$$

$$\int \frac{A\,dx}{(x-a)^n} \quad n \neq 1 \qquad \int \frac{A\,dx}{(x-a)^n} = A \cdot \frac{1}{1-n} \cdot \frac{1}{(x-a)^{n+1}} + C$$

2) $$\int \frac{Bx+C}{\left((x-\alpha)^2+\beta^2\right)^m}\,dx \qquad m = 1$$

$$t = \frac{x-\alpha}{\beta} \iff x = \beta t + \alpha \quad \text{donc} \quad \boxed{dx = \beta\, dt}$$

$$\frac{Bx+C}{\left((x-\alpha)^2+\beta^2\right)^m} = \frac{B(\beta t+\alpha)+C}{\left((\beta t)^2+\beta^2\right)^m} = \frac{B(\beta t+\alpha)+C}{\beta^{2m}\,(t^2+1)^m}$$

$$\frac{Bx+C}{\left((x-\alpha)^2+\beta^2\right)^m}\,dx = \frac{B(\beta t+\alpha)+C}{\beta^{2m-1}(t^2+1)^m} \times 1\, dt$$

$$= \frac{1}{\beta^{2m-1}} \frac{B(\beta t+\alpha)}{(t^2+1)^m}\,dt + \frac{C}{\beta^{2m-1}\,(t^2+1)^m}\,dt$$

$$= \frac{1}{\beta^{2m-2}} \frac{B\,t\,dt}{(t^2+1)^m} + \frac{\alpha B}{\beta^{2m-1}(t^2+1)^m}\,dt + \frac{C}{\beta^{2m-1}\,(t^2+1)^m}\,dt$$

$$= \frac{B}{2\beta^{2m-2}} \int \frac{(t^2+1)'}{(1+t^2)^m} + \frac{B\alpha+C}{\beta^{2m-1}} \int \frac{dt}{(1+t^2)^m}$$