Séance 1 : Les intégrales de Riemann et calcul- Exercices

mercredi 1 mars 2023 19:24

Exercice 1

1)

$$u_n = \frac{1}{n}\left(\sin\left(\frac{\pi}{n}\right) + \sin\left(\frac{2\pi}{n}\right) + \ldots + \sin\left(\frac{n\pi}{n}\right)\right)$$

2)

$$t_n = n\left(\frac{1}{(n+1)^2} + \ldots + \frac{1}{(n+n)^2}\right)$$

3)

$$v_n = \frac{\sqrt{1} + \sqrt{2} + \ldots + \sqrt{n-1}}{n\sqrt{n}}$$

4)

$$w_n = \sqrt[n]{\left(1 + \left(\frac{1}{n}\right)^2\right)\left(1 + \left(\frac{2}{n}\right)^2\right)\ldots\left(1 + \left(\frac{n}{n}\right)^2\right)}$$

5)

$$v_n = \frac{1}{n}\prod_{k=1}^{n}(k+n)^{1/n}.$$

6)

$$S_n = \sum_{p=n}^{2n}\frac{1}{p}$$

7)

$$u_n = \sum_{k=1}^{n}\frac{n}{n^2+k^2}$$

8)

$$v_n = \sum_{k=n}^{2n-1}\frac{1}{n+k}$$

9)

$$w_n = \frac{1}{n}\sqrt[n]{\frac{(2n)!}{n!}}$$

$$\sum_{k=0}^{n} \frac{1}{n} f(x_k) \xrightarrow[n\to+\infty]{} \int_0^1 f(x)\,dx \quad \left(x_k = \frac{k}{n}\right)$$

1) $u_n = \frac{1}{n} \sum_{0}^{n-1} \sin\left(\frac{k\pi}{n}\right)$

$$\xrightarrow[n \to +\infty]{} \int_0^1 \sin(\pi x)\,dx = \left[-\frac{\cos(\pi x)}{\pi}\right]_0^1$$

$$= \frac{2}{\pi} \qquad\qquad x_k = \frac{k}{n}$$

2) $t_n = n \sum_{1}^{n} \frac{1}{(n+k)^2}$

$$= \cancel{n} \sum \frac{1}{n^2\left(1+\frac{k}{n}\right)^2} = \frac{1}{n} \cdot \sum_{1}^{n} \frac{1}{\left(1+\frac{k}{n}\right)^2}$$

$$= \frac{1}{n} \cdot \sum_{1}^{n} \frac{1}{(1+x_k)^2}$$

$$\xrightarrow[n \to +\infty]{} \int_0^1 \frac{1}{(1+x)^2}\,dx = \left[-\frac{1}{1+x}\right]_0^1$$

$$= 1/2 .$$

3) $v_n = \frac{\sqrt{1} + \sqrt{2} + \dots + \sqrt{n-1}}{n\sqrt{n}}$

$$= \frac{1}{n} \sum_{n=1}^{n-1} \sqrt{\frac{k}{n}}$$

$$\xrightarrow[n \to +\infty]{} \int_0^1 \sqrt{x}\,dx = 2/3 .$$

4) $w_n = \sqrt[n]{\dfrac{\left(1+\left(\frac{1}{n}\right)^2\right)\left(1+\left(\frac{2}{n}\right)^2\right) \cdots \left(1+\left(\frac{n}{n}\right)^2\right)}{n}}$

$$\ln \omega_n = \frac{1}{n} \sum_{k=1} \ln\left(1+\left(\frac{k}{n}\right)^2\right)$$

$$\ln \prod = \sum \ln$$

$$\ln \omega_n \xrightarrow[n\to+\infty]{} \int_0^1 1\times \ln(1+x^2)\,dx$$

$$= \ln(2) - 2 + \frac{\pi}{2}$$

Comme $x \mapsto e^x$ est continue :

$$\ln \omega_n = \ln(2) - 2 + \frac{\pi}{2}$$

$$\boxed{\omega_n = 2 \cdot e^{\frac{\pi}{2}-2}}$$

4) $$v_n = \frac{1}{n} \prod_{k=1}^{n} (k+n)^{1/n}$$

$$\ln v_n = -\ln(n) + \frac{1}{n}\sum_{k=1}^{n} \ln(k+n)$$

$$= -\ln(n) + \frac{1}{n} \sum_k \left(\ln(n) + \ln\left(1+\frac{k}{n}\right)\right)$$

$$= -\ln(n) + \frac{1}{n}\sum_{k=1}^{n} \ln(n) + \frac{1}{n}\sum \ln\left(1+\frac{k}{n}\right)$$

$$= \cancel{-\ln(n)} + \cancel{\frac{1}{n}} \times \cancel{\ln(n) \times n} + \frac{1}{n}\sum \ln\left(1+\frac{k}{n}\right)$$

$$= \frac{1}{n}\sum_{1}^{n} \ln\left(1+\frac{k}{n}\right)$$

$$\xrightarrow[+\infty]{} \int_0^1 1\times \ln(1+x)\,dx = 2\ln 2 - 1$$

$$\ln v_n \longrightarrow 2\ln(2) - 1$$

$$u_n \longrightarrow e^{2\ln 2 - 1} = \frac{4}{e}$$

6) $S_n = \sum_{p=n}^{2n} \frac{1}{p}$

$$= \sum_{k=0}^{n} \frac{1}{k+n} \qquad (p = k+n)$$

$$= \frac{1}{n} \sum_{0}^{n} \frac{1}{1+\frac{k}{n}}$$

$$\xrightarrow[n\to+\infty]{} \int_0^1 \frac{1}{1+x} dx = \left[\ln(1+x)\right]_0^1$$

$$= \ln(2)$$

7) $w_n = \frac{1}{n} \sqrt[n]{\frac{(2n)!}{n!}}$

$$\frac{(2n)!}{n!} = \frac{\cancel{1}\times\cancel{2}\times\cdots\times\cancel{n}\times\cdots\times 2n}{1\times 2\times\cdots\times n} = (n+1)\cdots(n+n)$$

$$= \prod_{k=1}^{n} (n+k)$$

$$w_n = \frac{1}{n} \prod_{k=1}^{n} (n+k)^{1/n}$$

$$\ln(w_n) = -\ln(n) + \frac{1}{n} \sum_{k=1}^{n} \ln(n+k)$$

$$= \frac{1}{n} \sum_{1}^{n} \left(\ln(n+k) - \ln(n)\right)$$

$$= \frac{1}{n} \sum_{1}^{n} \ln\left(\frac{n+k}{n}\right)$$

$$= \frac{1}{n} \sum_{1}^{n} \ln\left(\frac{n+k}{n}\right)$$

$$= \frac{1}{n} \sum_{1}^{n} \ln\left(1 + \frac{k}{n}\right)$$

$$\longrightarrow \int_0^1 \ln(1+x) = 2\ln(2) - 1$$

$$\omega_n \to \frac{4}{e}$$

Exercice 2

Calculer $\int_0^1 x^2 \, dx$ et $\int_0^1 x^3 \, dx$ en utilisant des sommes de Riemman.

$$\int_0^1 x^2 \, dx = \lim_{n \to +\infty} \frac{1}{n} \sum_{1}^{n} \left(\frac{k}{n}\right)^2$$

$$\sum k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$= \left(\lim_{n \to +\infty} \frac{1}{n^3} \left(\sum_{1}^{n} k^2\right)\right)$$

$$= \lim_{n \to +\infty} \frac{1}{n^2} \times \frac{(n+1)(2n+1)}{6} = \frac{1}{3}$$

$$\int_0^1 x^3 \, dx$$ à faire

Exercice 9

Soit $x \in \mathbb{R} \backslash \{-1, 1\}$, on pose $f(x) = \int_0^{2\pi} \ln\left(x^2 - 2x\cos t + 1\right) dt$.

1) Déterminer Df.
2) Factoriser sur $\mathbb{C}$ le polynôme $X^n - 1$.
3) Calculer $f(x)$ à l'aide de ses sommes de Riemann.