Intégrale de Wallis $\quad W_n = \displaystyle\int_0^{\pi/2} \sin^n(t)dt.$

1. Montrer que $W_n = \int_0^{\pi/2} \cos^n(t)dt$ ✓
2. Calculer W_0, W_1 et W_2
3. Démontrer que la suite (W_n) est décroissante.
4. Démontrer que, pour $n \geq 1$, on a $W_{n+1} = \frac{n}{n+1} W_{n-1}$.
5. Démontrer que pour tout $n \geq 1$, on a $nW_nW_{n-1} = \frac{\pi}{2}$.
6. Démontrer que, pour $p \in \mathbb{N}$ on a $W_{2p} = \frac{(2p)!}{2^{2p}(p!)^2}\frac{\pi}{2}$ et $W_{2p+1} = \frac{2^{2p}(p!)^2}{(2p+1)!}$.
7. Démontrer que $\lim \frac{W_{n+1}}{W_n} = 1$. En déduire que $W_n \sim \sqrt{\frac{\pi}{2n}}$.
8. En déduire un équivalent du coefficient binomial : $\binom{2p}{p} \sim 2^{2p}\sqrt{\frac{1}{p\pi}}$.

1) $W_n = \int_0^{\frac{\pi}{2}} \cos^n(t)\,dt$: soit $x = \frac{\pi}{2} - t$ $\qquad dx = -dt$.

$$W_n = -\int_{\frac{\pi}{2}}^{0} \sin^n\left(\frac{\pi}{2} - x\right) dx = \int_0^{\frac{\pi}{2}} \cos^n(x)\,dx$$

$t = x$: $\quad W_n = \int_0^{\pi/2} \cos^n(t)\,dt$

2) W_0, W_1, W_2 :

$$W_0 = \int_0^{\pi/2} dt = \frac{\pi}{2}$$

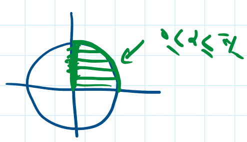
$$W_1 = \int_0^{\pi/2} \sin(t)\,dt = -\left[\cos(t)\right]_0^{\pi/2} = +1$$

$$W_2 = \int_0^{\pi/2} \sin^2(t)\,dt = \int_0^{\pi/2} \frac{1-\cos(2t)}{2}\,dt$$

$$= \frac{1}{2}\left\{\int_0^{\pi/2} dt - \int_0^{\pi/2} \cos(2t)\,dt\right\} = \frac{1}{2}\left\{[t]_0^{\pi/2} - \left[\frac{\sin(2t)}{2}\right]_0^{\pi/2}\right.$$

$$= \frac{1}{2}\left\{\frac{\pi}{2} - \frac{1}{2}\times 0\right\} = \frac{\pi}{4}$$

3) W_n est décroissante : $\quad W_{n-1}\ W_n \quad (W_n\ \ W_{n+1})$



$0 \leq \sin(t) \leq 1 \qquad \forall t \in [0, \frac{\pi}{2}]$

$0 \leq \sin^n(t) \leq \sin^{n-1}(t) \qquad \forall t \in [0, \frac{\pi}{2}]$ $\qquad 0 \leq \frac{1}{2} \leq 1$

$\int_0^{\frac{\pi}{2}} \sin^n(t)\,dt \leq \int_0^{\frac{\pi}{2}} \sin^{n-1}(t)\,dt$ $\qquad \frac{1}{4}$

$W_n \leq W_{n-1}$ alors W_n est une suite décroissante.

4) $W_{n+1} = \frac{n}{\ldots} W_{n-1}$: relation de récurrence :

4) $W_{n+1} = \frac{n}{n+1} W_{n-1}$: relation de récurrence :

$$W_{n+1} = \int_0^{\pi/2} \sin^{n+1}(t)\,dt = \int_0^{\pi/2} \underbrace{\sin^{n}(t)}_{v} \cdot \underbrace{\sin(t)}_{u'}\,dt .$$

$$\begin{cases} u'(t) = \sin(t) \\ v(t) = \sin^n(t) \end{cases} \rightarrow \begin{cases} u(t) = -\cos(t) \\ v'(t) = n\cos(t).\sin^{n-1}(t) \end{cases}$$

$$W_{n+1} = \Big[-\cos(t).\sin^n(t)\Big]_0^{\pi/2} + n\int_0^{\pi/2} \cos^2(t)\,\sin^{n-1}(t)\,dt$$

$$= 0 + n\int_0^{\pi/2} \left(1-\sin^2(t)\right) \sin^{n-1}(t)\,dt$$

$$= n\int_0^{\pi/2} \sin^{n-1}(t)\,dt - n\underbrace{\int_0^{\pi/2} \sin^{n+1}(t)\,dt}_{W_{n+1}}$$

$$W_{n+1} = n\left(W_{n-1} - W_{n+1}\right)$$

$$W_{n+1}(n+1) = n\,W_{n-1} \Rightarrow W_{n+1} = \frac{n}{n+1} W_{n-1}$$

5) $n\,W_n \cdot W_{n-1} = \frac{\pi}{2}$ $\forall n \geq 1$:

$$\frac{W_{n+1}}{W_{n-1}} = \frac{n}{n+1}$$

$n \to n-1$: $\frac{W_n}{W_{n-2}} = \frac{n-1}{n}$: D'après (4)

$$\frac{W_n}{\cancel{W_{n-2}}} \times \frac{W_{n-1}}{\cancel{W_{n-3}}} \times \frac{\cancel{W_{n-2}}}{\cancel{W_{n-4}}} \times \cdots \times \frac{W_2}{W_0} = \frac{\cancel{n-1}}{n} \times \frac{\cancel{n-2}}{\cancel{n-1}} \times \frac{\cancel{n-3}}{\cancel{n-2}} \times \cdots \times \frac{1}{\cancel{2}}$$

$$W_n \times W_{n-1} = W_0\left(\frac{1}{n}\right)$$

$$n.W_n.W_{n-1} = \frac{\pi}{2}. \quad \forall n \geq 1.$$

6) Mq : $\forall p \in \mathbb{N}$: $W_{2p} = \frac{(2p)!}{2^{2p}(p!)^2} \cdot \frac{\pi}{2}$, $W_{2p+1} = \frac{2^{2p}.(p!)^2}{(2p+1)!}$

6) Mq : $\forall p \in \mathbb{N}$: $W_{2p} = \frac{(2p)!}{2^{2p}\cdot(p!)^2}\cdot\frac{\pi}{2}$, $W_{2p+1} = \frac{2^{2p}\cdot(p!)^2}{(2p+1)!}$

d'après : (4) : $W_{n+1} = \frac{n}{n+1}\cdot W_{n-1}$

• $W_n = \frac{n-1}{n} W_{n-2}$

$n = 2p$: $W_{2p} = \frac{2p-1}{2p}\times W_{2p-2} = \frac{2p-1}{2p}\times\frac{2p-3}{2p-2}\times W_{2p-4} = \frac{2p-1}{2p}\times\frac{2p-3}{2p-2}\times\dots\frac{1}{2}\times W_0$

$$W_{2p} = \frac{(2p-1)(2p-3)\dots\dots 1}{(2p)(2p-2)\times\dots\dots\times 2} = \frac{2p}{(2p)(2p)}\frac{(2p-1)(2p-2)}{(2p-2)^2}\frac{(2p-3)(2p-4)}{(2p-4)^2}\dots\times\frac{1}{2}$$

$$= \frac{(2p)!}{2^2(p)^2 2(p-1)^2}\times\dots\frac{1}{2^2}.$$

$$\boxed{W_{2p} = \frac{(2p)!}{2^{2p}p!}}$$

$n = 2p+1$:

$$\frac{2^{2p}\cdot(p!)^2}{(2p+1)!}$$

$$W_{2p+1} = \frac{2p}{2p+1} W_{2p-1} = \frac{2p}{2p+1}\times\frac{2p-2}{2p-1}\times W_{2p-3}$$

$$= \frac{2p}{2p+1}\times\frac{2p-2}{2p-1}\times\dots\times\frac{2}{3}\cdot W_1$$

$n \geq 1$

$2p+1 \geq 3$.

$$= \frac{(2p)^2 \quad (2p-2)^2\times\dots}{(2p+1)\times 2p\times(2p-1)(2p-2)\times\dots} = \frac{2^{2p}\cdot(p!)^2}{(2p+1)!}$$

$$W_{2p+1} = \frac{2^{2p}\cdot(p!)^2}{(2p+1)!}$$

7) $\lim_{+\infty} \frac{W_{n+1}}{W_n} = 1$

On a W_n est décroissante. $W_{n+2} \leq W_{n+1} \leq W_n$ $(W_n \geq 0)$

$$\frac{W_{n+2}}{W_n} \leq \frac{W_{n+1}}{W_n} \leq 1$$

$$\frac{n+1}{n+2} \leq \frac{W_{n+1}}{W_n} \leq 1$$

$\downarrow$ 1 $\qquad\qquad \downarrow$ 1

$W_{n+1} = \frac{n}{n+1}\cdot W_{n-1}$

$\downarrow$ n+1

$$\frac{W_{n+2}}{W_n} = \frac{n+1}{n+2}$$

$\lim_{+\infty}\frac{W_{n+1}}{W_n} = 1$ ✓ * *

$\underset{+\infty}{\sim} \quad W_n$

* $W_n \sim \left(\frac{\pi}{2n}\right)^{1/2}$

$n \cdot W_n W_{n-1} = \frac{\pi}{2}$ (Q.T)

$$W_n^2 \left(\frac{W_{n-1}}{W_n}\right) = \left(\frac{\pi}{2n}\right)$$

$$\frac{W_{n-1}}{W_n} \to 1$$

$$W_n^2 \underset{+\infty}{\sim} \left(\frac{\pi}{2n}\right)$$

$$\boxed{W_n \sim \left(\frac{\pi}{2n}\right)^{1/2}}$$

8) En déduire que : $\binom{2p}{p} \sim 2^{2p} \left(\frac{1}{p \cdot \pi}\right)^{1/2}$

D'après (6) : $W_{2p} = \frac{(2p)!}{2^{2p}(p!)^2} \cdot \frac{\pi}{2} = \binom{2p}{p} \cdot \frac{1}{2^{2p}} \cdot \frac{\pi}{2}$ *

D'après (7) : $W_{2p} \sim \left(\frac{\pi}{4p}\right)^{1/2}$

$$\binom{2p}{p} = \frac{(2p)!}{(2p-p)!\,p!} = \frac{(2p)!}{(p!)^2}$$

$$\binom{2p}{p} \cdot \frac{1}{2^{2p}} \cdot \frac{\pi}{2} \sim \left(\frac{\pi}{4p}\right)^{1/2}$$

$$\binom{2p}{p} \sim 2^{2p} \frac{1}{\sqrt{p \cdot \pi}}$$