Séance 2 - Thermodynamique - Exercices

vendredi 21 octobre 2022 22:12

❖ Formes différentielles et intégration – Equation d'état

1) Calculer la différentielle de la fonction : $f(x,y) = \frac{x^2}{y} + \frac{x}{y^2} - 2x^3y^2$

2) On considère les formes différentielles suivantes:

$$\delta f = \frac{x^2}{y^2}dx - \frac{2}{3}\frac{x^3}{y^3}dy$$

$$\delta g = \left(\frac{x^2}{y} - 2y\right)dx - \left(\frac{x^3}{y^2} + 3x\right)dy$$

a) Laquelle de ces deux formes est une différentielle totale?
b) Déterminer alors la fonction correspondante.

3) Montrer que la forme différentielle suivante est exacte:

$$\delta w = -\frac{RT}{V^2}\left(1 + \frac{2A}{V}\right)dV + \frac{R}{V}\left(1 + \frac{A}{V}\right)dT$$

où A et R sont des constantes.

δw représente en fait la différentielle dP de la pression d'un gaz, en déduire alors l'équation d'état de ce gaz

❖ Fonctions implicites

(L'exercice N°4 ne sera pas traité en TD, veuillez bien de le faire à la maison)

4) On considère l'équation d'état d'un système $f(P,V,T)=0$, où P, V, et T sont les paramètres d'état du système.
a) Donner les expressions de dP et dV.
b) En déduire les relations suivantes:

$$\left(\frac{\partial P}{\partial V}\right)_T\left(\frac{\partial V}{\partial P}\right)_T = 1 \quad et \quad \left(\frac{\partial P}{\partial V}\right)_T\left(\frac{\partial V}{\partial T}\right)_P\left(\frac{\partial T}{\partial P}\right)_V = -1$$

c) Vérifier ces relations pour une mole de gaz parfait, d'équation d'état: $PV=RT$.

1°. $df = \left.\frac{\partial f}{\partial x}\right|_y dx + \left.\frac{\partial f}{\partial y}\right|_x dy$: forme diff.

$\left.\frac{\partial f}{\partial x}\right|_y = \left.\frac{\partial}{\partial x}\right|_y \left(\frac{x^2}{y} + \frac{x}{y^2} - 2x^3 \cdot y^2\right)$. $\quad (f+g+h)' = f'+g'+h'$

$= \frac{\partial}{\partial x}\left(\frac{x^2}{y}\right) + \frac{\partial}{\partial x}\left(\frac{x}{y^2}\right) - \frac{\partial}{\partial x}(2x^3 \cdot y^2)$ $\quad (\alpha f)' = \alpha \cdot f'$

$= \frac{1}{y} \cdot \frac{\partial}{\partial x}(x^2) + \frac{1}{y^2} \cdot \frac{\partial}{\partial x}(x) - 2y^2 \cdot \frac{\partial}{\partial x}(x^3)$.

$$\boxed{\left.\frac{\partial f}{\partial x}\right|_y = \frac{1}{y} 2 \cdot x + \frac{1}{y^2} - 2y^2 \cdot 3 \cdot x^2.}$$

$\left.\frac{\partial f}{\partial y}\right|_x = x^2 \cdot \left(-\frac{1}{y^2}\right) + x \cdot \left(-\frac{2}{y^3}\right) - 2x^3 \cdot 2y$.

$\left.\frac{\partial f}{\partial y}\right|_x = -\frac{x^2}{y^2} - \frac{2x}{y^3} - 4x^3 \cdot y$.

$$df = \left(\frac{2x}{y} + \frac{1}{y^2} - 6y^2 x^2\right)dx - \left(\frac{x^2}{y^2} + \frac{2x}{y^3} + 4yx^3\right)dy$$

2. Condition Schwartz : $\delta f = \boxed{P} \cdot \boxed{dx} + \boxed{Q} \cdot \boxed{dy}$

$$\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}.$$

δf : total exacte.
$\delta f = df$.

$\delta f = \frac{x^2}{y^2}dx - \frac{2}{3} \cdot \frac{x^3}{y^3} \cdot dy$.

$P(x,y) = \frac{x^2}{y^2}$, $Q(x,y) = -\frac{2}{3} \cdot \frac{x^3}{y^3}$.

$\left.\frac{\partial P}{\partial y}\right|_x = x^2 \cdot \left(-\frac{2}{y^3}\right) = -\frac{2x^2}{y^3}$

$\left.\frac{\partial Q}{\partial x}\right|_y = -\frac{2}{\cancel{3}} \cdot \frac{1}{y^3}(\cancel{3} \cdot x^2) = -\frac{2x^2}{y^3}$.

$$\boxed{\left.\frac{\partial P}{\partial y}\right|_x = \left.\frac{\partial Q}{\partial x}\right|_y}$$

δf : est totale exacte
$\delta f = df$.

$$\boxed{\left.\frac{\partial P}{\partial y}\right|_x = \left.\frac{\partial Q}{\partial x}\right|_y}$$

δf : est totale exacte

$\delta f = df$.

* $\delta g = \left(\frac{x^2}{y} - 2y\right)dx - \left(\frac{x^3}{y^2} + 3x\right)dy$.

$$\left.\frac{\partial}{\partial y}\left(\frac{x^2}{y} - 2y\right)\right|_x = x^2\cdot\left(-\frac{1}{y^2}\right) - 2 = -\left(\frac{x^2}{y^2}+2\right)$$

$$\left.\frac{\partial}{\partial x}\left(-\frac{x^3}{y^2} - 3x\right)\right|_y = -\frac{1}{y^2}\cdot 3x^2 - 3 = -\left(3\frac{x^2}{y^2}+3\right)$$

$$\boxed{\left.\frac{\partial P}{\partial y}\right|_x \neq \left.\frac{\partial Q}{\partial x}\right|_y}$$: δg : n'est pas totale exacte.

b) $df = \frac{x^2}{y^2}dx - \frac{2}{3}\frac{x^3}{y^3}\cdot dy \rightsquigarrow f(x,y)$

$$\begin{cases} \frac{\partial f}{\partial x} = \frac{x^2}{y^2} & (E_1) \\ \frac{\partial f}{\partial y} = -\frac{2}{3}\cdot\frac{x^3}{y^3} & (E_2). \end{cases}$$

Etape 1 : (E_1) : $\begin{cases} \frac{df}{dx} = \frac{x^2}{y^2} & \rightarrow \int df = \int \frac{x^2}{y^2}\cdot dx. \\ y = \text{cte}. & \rightarrow f(x,y) = \frac{1}{y^2}\cdot\frac{x^3}{3} + C(y) \quad (i) \end{cases}$

Etape 2 : $C(y)$

(i) $\frac{\partial f}{\partial y} = -\frac{2}{3}\cdot\frac{x^3}{y^3} + \frac{\partial C}{\partial y}$. (E_3)

$E_3 = E_2$: $-\frac{2}{3}\cdot\cancel{\frac{x^3}{y^3}} + \frac{\partial C}{\partial y} = -\frac{2}{3}\cdot\cancel{\frac{x^3}{y^3}}$

$$\downarrow$$

$$\left\|\frac{\partial C(y)}{\partial y} = 0\right\|$$

$$C = k \in \mathbb{R}.$$

(i) $$\boxed{f(x,y) = \frac{x^3}{3y^2} + k}$$

3) $\delta w = \underbrace{-\frac{RT}{V^2}\cdot\left(1+\frac{2A}{V}\right)}_{B(T,V)}dV + \underbrace{\frac{R}{V}\left(1+\frac{A}{V}\right)}_{C(T,V)}dT$.

• $\left.\frac{\partial B}{\partial T}\right|_V = \frac{\partial}{\partial T}\cdot\left\{-\frac{RT}{V^2}\left(1+\frac{2A}{V}\right)\right\}\Big|_V$

$$= -\frac{R}{V^2}\cdot\left(1+\frac{2A}{V}\right)\frac{\partial T}{\partial T} = -\frac{R}{V^2}\cdot\left(1+\frac{2A}{V}\right)$$

• $\left.\frac{\partial C}{\partial V}\right|_T = R\cdot\frac{\partial}{\partial V}\left(\frac{1}{V} + \frac{A}{V^2}\right) = -\frac{R}{V^2} - \frac{2AR}{V^3}$

$$= -\frac{R}{V^2}\cdot\left(1+\frac{2A}{V}\right).$$

$$\boxed{\left.\frac{\partial B}{\partial T}\right|_V = \left.\frac{\partial C}{\partial V}\right|_T}$$

$\delta w = dw$: D.T.E.

$= dP$.

$dP = -\frac{RT}{V^2}\cdot\left(1+\frac{2A}{V}\right)dV + \frac{R}{V}\cdot\left(1+\frac{A}{V}\right)dT$.

$\hookrightarrow P(T,V) = P$

Etape : 1 : $dP = \left.\frac{\partial P}{\partial T}\right|_V\cdot dT + \left.\frac{\partial P}{\partial V}\right|_T\cdot dV$. (i)

$$\begin{cases} \left.\dfrac{\partial P}{\partial V}\right|_T = -\dfrac{RT}{V^2}\left(1+\dfrac{2A}{V}\right) & (E_1) \\ \left.\dfrac{\partial P}{\partial T}\right|_V = \dfrac{R}{V}\left(1+\dfrac{A}{V}\right) & (E_2) \end{cases}$$

$$\begin{cases} \dfrac{dP}{dT} = \dfrac{R}{V}\left(1+\dfrac{A}{V}\right) & (1) \\ \| V = \text{cte} \| \end{cases}$$

$$(1) \quad dP = \frac{R}{V}\left(1+\frac{A}{V}\right) \cdot dT$$

$$P = \frac{R}{V}\left(1+\frac{A}{V}\right)T + f(V) \qquad (I)$$

Etape 2 : (I) $\left.\dfrac{\partial P}{\partial V}\right|_T = -\dfrac{RT}{V^2}\left(1+\dfrac{2A}{V}\right) + \left.\dfrac{\partial f}{\partial V}\right|_T$

$$= -\frac{RT}{V^2}\left(1+\frac{2A}{V}\right) \qquad (E_2)$$

$$\left.\frac{\partial f}{\partial V}\right|_T = 0 \Rightarrow f = \text{cte} = k \in \mathbb{R}.$$

$$\boxed{P = \frac{R}{V}\left(1+\frac{A}{V}\right)T + k}$$

* à très faible pression. G.R → G.P

$P.V \to 0$ qd $T \to 0$ (i)

$PV = R\left(1+\frac{A}{V}\right)T + k$

$\| k = 0 \|$

$$\boxed{PV = R\left(1+\frac{A}{V}\right)T}$$

Exercice N°1

Calculer le travail échangé lors de la compression isotherme d'une mole de gaz parfait de l'état (P_1 = 2 atm, T_1 = 300 K) à l'état $P_2 = 5\, P_1$, dans les trois cas suivants:

1) Transformation réversible.
2) Compression brusque de P_1 à P_2.
3) Compression brusque de P_1 à $P_3 = 2\, P_2$ puis détente de P_3 à P_2.

On donne: $R = 8{,}32\ J.K^{-1}$.

E.I: $P_1 = 2\text{atm}$, $T_1 = 300K$, V_1 $\xrightarrow{\text{isotherme}}$ E.F: $P_2 = 5.P_1$, $T_2 = T_1 = 300k$, V_2

P_{ext} ; P_{ext}

1) Transformation reversible (Lente) : $\Rightarrow P_{ext} = P.$ (pression du gaz)

$$GP\,(n=1) : \begin{cases} \delta W_{rev} = -P_{ext}\cdot dV = -P\,dV. \\ P = \dfrac{RT}{V} = \dfrac{RT_1}{V} \end{cases}$$

$$W_{rev} = \int \delta W_{rev} = \int_1^2 -P.dV.$$

$P = \frac{RT}{V} = \frac{RT_1}{V}$

$$W_{rev} = \int \delta W_{rev} = \int_1^2 -P.dV.$$

$$W_{rev} = -RT_1 \int_{V_1}^{V_2} \frac{dV}{V} = -R.T_1.\ln\frac{V_2}{V_1}$$

$$E.I \begin{cases} P_1 \\ T_1 \\ V_1 \end{cases} \xrightarrow{\text{isotherme}} E.F \begin{cases} P_2 = r P_1 \\ T_1 \\ V_2 \end{cases}$$

$P.V = RT = \underbrace{cte}$ (isotherme).

$(PV)_i = (P.V)_f \iff P_1.V_1 = P_2.V_2$

$\iff \frac{V_2}{V_1} = \frac{P_1}{P_2} = \frac{P_1}{rP_1} = \frac{1}{r}$

$$\boxed{W_{rev} = -R.T_1.\ln\left(\frac{1}{r}\right) = +R.T_1.\ln(r).}$$

AN: $W_{rev} = 4017\,J.$

2) brusque : (irréversible) rapide

$\Rightarrow \delta W_{irre} = -P_{ext}.dV. = -P_2.dV$

$\Rightarrow W_{irre} = -P_2.\int_{V_1}^{V_2} dV = -P_2.(V_2 - V_1).$

$P.V = cte \Rightarrow P_1V_1 = P_2V_2 = R.T_1.$

$$\begin{cases} V_1 = \frac{RT_1}{P_1} \\ V_2 = \frac{RT_1}{P_2} \end{cases}$$

$W_{irrev} = -P_2.R.T_1\left\{\frac{1}{P_2} - \frac{1}{P_1}\right\}.$

$$\boxed{W_{irr.} = -R.T_1\left\{1 - \frac{P_2}{P_1}\right\}.}$$

AN: $W_{irr} = 9984\,J.$

$$\boxed{W_{irrev} > W_{rev}.}$$

3) Compression brusque : $P_1 \longrightarrow P_3 = 2P_1$

$P_3 \longrightarrow P_2$

$$E_1. \begin{cases} P_1 \\ T_1 \\ V_1 \end{cases} \xrightarrow[P_{ext} = P_3]{} E_2 \begin{cases} P_3 = 2P_1 \\ T_3 \\ V_3 \end{cases} \xrightarrow[P_{ext} = P_2]{} E_3 \begin{cases} P_2 = rP_1 \\ T_2 \\ V_2. \end{cases}$$

$W = W_{12} + W_{23} = -P_3.(V_3 - V_1) - P_2(V_2 - V_3)$

$$W = P_2.(-V_3 + 2V_1 - V_2) = \cancel{P_2}.\left(-\frac{RT_1}{2\cancel{P_2}} + 2.r\frac{RT_1}{\cancel{P_2}} - \frac{RT_1}{\cancel{P_2}}\right)$$

$T_1 = T - T_3$ et $P_1 = \frac{1}{r} P_2$

E_1: $P_1V_1 = RT_1 \rightarrow V_1 = \frac{RT_1}{P_1} = r.\frac{RT_1}{P_2}$

E_2 : $P_3 \cdot V_3 = RT_1 \rightarrow V_3 = \frac{RT_1}{P_3} = \frac{RT_1}{2P_2}$

E_3 : $P_2 V_2 = RT_1 \rightarrow V_2 = \frac{RT_1}{P_2}$

$\boxed{W = 21\,216 \text{ J}/1 \quad AN}$