OFFRE BE IN SCIENCES/ COURS DE SOUTIEN

ELECTROSTATIQUE

Les opérateurs différentiels

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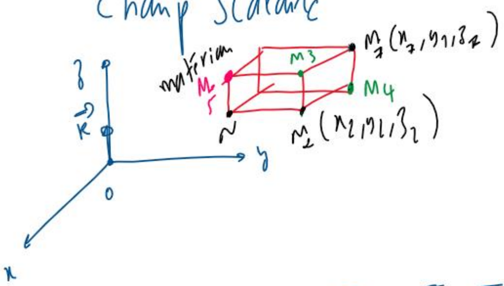
Champ scalaire



$\vec{OM_1}\begin{vmatrix} x_1 \\ y_1 \\ z_1 \end{vmatrix}$; $\vec{OM_2}\begin{vmatrix} x_2 \\ y_2 \\ z_2 \end{vmatrix}$ $\Rightarrow f(M) = f(x,y,z)$

$$\left.\begin{aligned} T(M_1) &= T(x_1,y_1,z_1) = T_1 \\ T(M_2) &= T(x_2,y_2,z_2) = T_2 \end{aligned}\right\} \text{Température}$$

Champ non uniforme $\Rightarrow$ dep (x,y,z)

$$\left.\begin{aligned} M_1 \to M_4 &: \frac{\Delta T}{\Delta z} = \frac{T_4 - T_1}{z_4 - z_1} \longrightarrow \frac{\partial T}{\partial z}\vec{k} \\ M_4 \to M_2 &: \frac{\Delta T}{\Delta x} = \frac{T_2 - T_4}{x_2 - x_4} \longrightarrow \frac{\partial T}{\partial x}\vec{i} \\ N \to M_2 &\longrightarrow \frac{\partial T}{\partial y}\vec{j} \end{aligned}\right\} \quad \vec{\nabla} = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

$$\boxed{\vec{\nabla} f = \vec{\text{grad}}\, f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}}$$

Ex $f_1(x,y,z) = \sqrt{x^2+y^2+z^2} = r$

$\vec{OM} = x\vec{i} + y\vec{j} + z\vec{k} = r\vec{e_r}$

$OM = \sqrt{x^2+y^2+z^2} = r$

- $\vec{\nabla} f_1 = \vec{\text{grad}}\, f_1 = \frac{\partial f_1}{\partial x}\vec{i} + \frac{\partial f_1}{\partial y}\vec{j} + \frac{\partial f_1}{\partial z}\vec{k}$

- $\frac{\partial f_1}{\partial x} = \frac{\partial}{\partial x}\left(\sqrt{x^2+y^2+z^2}\right) = \frac{2x}{2\sqrt{x^2+y^2+z^2}} = \frac{x}{\sqrt{x^2+y^2+z^2}}$ $\quad \left(\sqrt{U}\right)' = \frac{U'}{2\sqrt{U}}$

- $\frac{\partial f_1}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}}$; $\frac{\partial f_1}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}}$

$$\vec{\text{grad}}\, f_1 = \frac{1}{\sqrt{x^2+y^2+z^2}}\Big[\underbrace{x\vec{i} + y\vec{j} + z\vec{k}}_{\vec{OM}}\Big] = \frac{\vec{OM}}{\|\vec{OM}\|} = \vec{e_r}$$

$$\boxed{f_1 = r}$$

En coordonnées sphériques : $f(r,\theta,\varphi)$: $\vec{\nabla} f = \vec{grad} f = \frac{\partial f}{\partial r}\vec{e_r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\vec{e_\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\vec{e_\varphi}$

$\hookrightarrow$ si $f = f(r)$ ne dépend que de r $= \frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial \varphi} = 0$

$$\left[\vec{\nabla} f = \frac{\partial f}{\partial r}\vec{e_r}\right]$$

$$\left[f_1 = r \Rightarrow \vec{grad} f_1 = \frac{\partial f_1}{\partial r}\vec{e_r} = \frac{\partial}{\partial r}(r)\vec{e_r} = \vec{e_r}\right]$$

- $f_2(r) = \frac{1}{r} = \frac{1}{\sqrt{x^2+y^2+z^2}}$; $\left(\frac{1}{X}\right)' = -\frac{1}{X^2}$

- $\vec{\nabla} f_2 = \vec{grad} f_2 = \frac{\partial f_2}{\partial r}\vec{e_r} = \frac{\partial}{\partial r}\left[\frac{1}{r}\right]\vec{e_r} = -\frac{1}{r^2}\vec{e_r}$

- $\vec{\nabla} f_2 = \frac{\partial f_2}{\partial x}\vec{i} + \frac{\partial f_2}{\partial y}\vec{j} + \frac{\partial f_2}{\partial z}\vec{k}$

$\left\{ f_2 = \left(x^2+y^2+z^2\right)^{-1/2} \right.$; $\left(U^n\right)' = n\,U^{n-1}\,U'$

- $\frac{\partial f_2}{\partial x} = -\frac{1}{\cancel{2}}\left[x^2+y^2+z^2\right]^{-\frac{3}{2}} \cdot \cancel{2}x$

$$\frac{\partial f_2}{\partial x} = \frac{-x}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}$$

$$\frac{\partial f_2}{\partial y} = \frac{-y}{\left(x^2+y^2+z^2\right)^{3/2}} \; ; \quad \frac{\partial f_2}{\partial z} = \frac{-z}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$\vec{grad} f_2 = \frac{-1}{\left(x^2+y^2+z^2\right)^{3/2}}\left(x\vec{i} + y\vec{j} + z\vec{k}\right) \; ; \quad r^2 = x^2+y^2+z^2$$

$r = \left(x^2+y^2+z^2\right)^{1/2}$

$\hookrightarrow r^3 = \left(x^2+y^2+z^2\right)^{3/2}$

$$\boxed{\vec{grad} f_2 = \frac{-1}{r^3}\left(r\vec{e_r}\right) = \frac{-1}{r^2}\vec{e_r}}$$

• potentiel $V(M) = V(r,\theta) = \dfrac{qa\cos(\theta)}{4\pi\varepsilon_0 r^2}$ $\qquad \left(\dfrac{1}{x^2}\right)' = -\dfrac{2}{x^3}$

$\vec{\text{grad}}\, V(M) = \vec{\nabla} V = \dfrac{\partial V}{\partial r}\vec{e_r} + \dfrac{1}{r}\dfrac{\partial V}{\partial \theta}\vec{e_\theta}$

• $\dfrac{\partial V}{\partial r} = \dfrac{\partial}{\partial r}\left(\dfrac{qa\cos\theta}{4\pi\varepsilon_0 r^2}\right) = \dfrac{qa\cos\theta}{4\pi\varepsilon_0}\dfrac{\partial}{\partial r}\left(\dfrac{1}{r^2}\right) =$

$$\left[\dfrac{\partial V}{\partial r} = \dfrac{-2qa\cos\theta}{4\pi\varepsilon_0 r^3}\right]$$

• $\dfrac{\partial V}{\partial \theta} = \dfrac{qa}{4\pi\varepsilon_0 r^2}\dfrac{\partial}{\partial\theta}[\cos\theta] = \dfrac{-qa\sin\theta}{4\pi\varepsilon_0 r^2}$

$\vec{\nabla} V = \dfrac{-2qa\cos\theta}{4\pi\varepsilon_0 r^3}\vec{e_r} - \dfrac{qa\cos\theta}{4\pi\varepsilon_0 r^3}\vec{e_\theta}$

$$\left(\vec{E} = -\vec{\text{grad}}\, V\right)$$

champ vectoriel $\quad \vec{E} = E_x(x,y,z)\vec{i} + E_y(x,y,z)\vec{j} + E_z(x,y,z)\vec{k}$

$\left(\text{div}\,\vec{E}_{(M)} = \dfrac{\rho(M)}{\varepsilon_0}\right) \Rightarrow \text{div}\,\vec{E} = \dfrac{\partial E_x}{\partial x} + \dfrac{\partial E_y}{\partial y} + \dfrac{\partial E_z}{\partial z}$

$$\vec{U} = \underbrace{(xz^2)}_{U_x}\vec{i} + \underbrace{(x^2y - z^3)}_{U_y}\vec{j} + \underbrace{(2xy + y^2z)}_{U_z}\vec{k}$$

$$\text{div}\,\vec{U} = \dfrac{\partial U_x}{\partial x} + \dfrac{\partial U_y}{\partial y} + \dfrac{\partial U_z}{\partial z}$$
$$= z^2 + x^2 + y^2 = x^2 + y^2 + z^2 = r^2$$

• $\left(\vec{B} = \vec{\text{rot}}(\vec{A})\right)$: $\quad \vec{\text{rot}}\,\vec{E} = \vec{\nabla}\wedge\vec{E} = \begin{vmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{vmatrix} \wedge \begin{vmatrix} E_x(x,y,z) \\ E_y(x,y,z) \\ E_z(x,y,z) \end{vmatrix}$

$$\vec{\text{rot}}\,\vec{E} = \begin{vmatrix} \partial/\partial y & E_y \\ \partial/\partial z & E_z \end{vmatrix}\vec{i} - \begin{vmatrix} \partial/\partial x & E_x \\ \partial/\partial z & E_z \end{vmatrix}\vec{j} + \begin{vmatrix} \partial/\partial x & E_x \\ \partial/\partial y & E_y \end{vmatrix}\vec{k}$$

$$\vec{rot}\,\vec{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\vec{i} - \left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right)\vec{j} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\vec{k}$$

$$\vec{U} = \underbrace{(xz^2)}_{U_x}\vec{i} + \underbrace{(x^2y - z^3)}_{U_y}\vec{j} + \underbrace{(2xy + y^2z)}_{U_z}\vec{k}$$

$$\vec{rot}\,\vec{U} = (2x + 2yz + 3z^2)\vec{i} - (2y - 2xz)\vec{j} + (2xy)\vec{k}$$

par ex : $\underbrace{V(M) = V(r)}_{\text{ne dépend que de } r} \Rightarrow \vec{grad}(V(M)) = \frac{\partial V}{\partial r}\vec{e_r}$

$\vec{E} = E(r)\vec{e_r}$

- sphérique : $div\,\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2E(r)\right)$
- cylindrique : $div\,\vec{E} = \frac{1}{r}\frac{\partial}{\partial r}\left(rE(r)\right)$