Séance 8 : Correction Examen FSTBMb

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Question de cours (4 pts) :

a- Montrer le théorème d'Ehernfest donnant l'évolution de la valeur moyenne d'un opérateur :

$$\frac{d\langle A\rangle_{|\psi\rangle}}{dt} = \frac{1}{i\hbar}\langle [A, H]\rangle_{|\psi\rangle} + \left\langle \frac{\partial A}{\partial t}\right\rangle_{|\psi\rangle}$$

b- En déduire $\frac{d\langle P_x\rangle_{|\psi\rangle}}{dt}$ où Px est l'opérateur impulsion d'une particule en mouvement suivant un axe (Ox) dans un potentiel V(x). Faire une comparaison avec la mécanique classique.

$$\langle H\rangle_{|\psi\rangle} = \langle\psi|H|\psi\rangle$$

a) $\langle A\rangle_{|\psi\rangle} = \langle\psi|A|\psi\rangle.$

$$\frac{d}{dt}\left(\langle\psi|A|\psi\rangle\right) = \frac{d\langle\psi|}{dt}.A|\psi\rangle + \langle\psi|\frac{dA}{dt}|\psi\rangle + \langle\psi|A\frac{d|\psi\rangle}{dt}$$

$$-\frac{i}{\hbar}H|\psi\rangle = .\frac{\partial|\psi\rangle}{\partial t}$$

$$\left(H|\psi\rangle\right)^+ = \left(i\hbar\frac{\partial|\psi\rangle}{\partial t}\right)^+ \qquad \left(A|\varphi\rangle\right)^+ = \langle\varphi|A^+$$

$$\frac{i}{\hbar}\langle\psi|H^+ = \frac{\partial\langle\psi|}{\partial t}$$

$H = H^+$: H : observable.

$$\frac{d}{dt}\left(\langle\psi|A|\psi\rangle\right) = \frac{i}{\hbar}.\langle\psi|H.A|\psi\rangle + \langle\psi|\frac{\partial A}{\partial t}|\psi\rangle - \langle\psi|A.\frac{i}{\hbar}.H|\psi\rangle$$

$$= \frac{i}{\hbar}.\langle\psi|(HA - AH)|\psi\rangle + \langle\psi|\frac{\partial A}{\partial t}|\psi\rangle.$$

$$= \frac{i}{\hbar}.\langle\psi|[H,A]|\psi\rangle + \langle\psi|\frac{\partial A}{\partial t}|\psi\rangle.$$

$$\boxed{\frac{d}{dt}\langle A\rangle_{|\psi\rangle} = \frac{i}{\hbar}\langle [H,A]\rangle_{|\psi\rangle} + \left\langle\frac{\partial A}{\partial t}\right\rangle_{|\psi\rangle}} = \frac{1}{i\hbar}.\langle [A,H]\rangle_{|\psi\rangle} + \left\langle\frac{\partial A}{\partial t}\right\rangle_{|\psi\rangle}$$

b. $A = P_x$: $\frac{d}{dt}\langle P_x\rangle = \frac{1}{i\hbar}.\langle [P_x, H]\rangle_{|\psi\rangle} + \langle\psi|\frac{\partial P_x}{\partial t}|\psi\rangle$

$$[P_x, H] = \left[P_x, \frac{P_x^2}{2m} + V(x)\right]$$

i) $[A, B+C] = [A,B] + [A,C]$

$$[P_x, H] = \underbrace{\left[P_x, \frac{P_x^2}{2m}\right]}_{=0} + [P_x, V(x)].$$

ii) $[A, f(B)] = [A,B].\frac{\partial f}{\partial B}$

$$[p_x, H] = 0 + [p_x, X]\cdot\frac{\partial V(x)}{\partial x}$$

$$= -i\hbar\cdot\frac{\partial V}{\partial x}.$$

$$[\hat{x}, \hat{p}_x] = i\hbar\cdot\mathbb{1}$$

$$\frac{d}{dt}\langle p_x\rangle_{(1)} = -i\hbar\left\langle\frac{\partial V}{\partial x}\right\rangle\frac{1}{\cancel{i\hbar}} + \left\langle\frac{\partial p_x}{\partial t}\right\rangle$$

$$F_x = -\frac{\partial V}{\partial x}.$$

$$\frac{d}{dt}\langle p_x\rangle = \langle F_x\rangle + \underbrace{\left\langle\frac{\partial p_x}{\partial t}\right\rangle}_{0}$$

$$p = -i\hbar\vec{\nabla}$$

$$\left(p_x = -i\hbar\nabla_x \Rightarrow \frac{\partial p_x}{\partial t} = 0.\right)$$

$$\left\|\ \frac{d}{dt}\langle p_x\rangle = \langle F_x\rangle.\ \right\|.$$

$$A = \hat{x} : \quad \left\|\ \frac{d\langle\hat{x}\rangle}{dt} = \frac{1}{m}\cdot\langle\hat{p}\rangle\ \right\|$$

Exercice 1 (8 pts)

On considère un système physique dont l'espace des états est formé par les trois vecteurs kets orthonormés suivants: $|u_1\rangle$, $|u_2\rangle$ et $|u_3\rangle$. Dans cette base, l'opérateur hamiltonien H et un opérateur A sont définis dans cette base par les matrices suivantes :

$$H=\hbar\omega\begin{pmatrix}1 & -1 & 0\\ -1 & 1 & 0\\ 0 & 0 & -1\end{pmatrix},\ A=a\begin{pmatrix}0 & 4 & 0\\ 4 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$$

$A = A^\dagger \Rightarrow$ Val prop. orth.

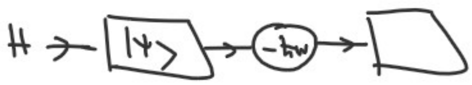
ω et a sont des constantes positives, ħ la constante de Planck.

1/ Calculer les valeurs propres de A et ses vecteurs propres normalisés. Vérifier qu'ils sont orthogonaux.

2/ Mêmes questions pour H. Le couple {A, H} forment ils un ECOC ?

3/ Supposons qu'à t=0, on mesure l'énergie. Quelles sont les valeurs possibles de cette mesure, à t=0 ? ✓

4/ Supposons qu'une mesure de l'énergie a donné la valeur - ħω. Immédiatement, après on mesure A, quelles valeurs obtient –t-on est avec quelles probabilités ?



$$\mathcal{E} = \{|u_1\rangle\ |u_2\rangle\ |u_3\rangle\}.$$

$$H = \hbar\omega.\begin{bmatrix}1 & -1 & 0\\ -1 & 1 & 0\\ 0 & 0 & -1\end{bmatrix} \qquad \hat{A} = a\begin{bmatrix}0 & 4 & 0\\ 4 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}$$

1. Val propre de $\hat{A}$:

$$\det(A - \lambda\,\mathbb{1}_{3,3}) = 0 \Rightarrow \begin{array}{c|ccc|} & + & - & + \\ \hline + & -\lambda & 4 & 0\\ - & 4 & -\lambda & 1\\ + & 0 & 1 & -\lambda \\ \hline \end{array} = 0$$

$$\det(A - \lambda \mathbb{1}_{3,3}) = 0 \Rightarrow \begin{array}{c} - \\ + \end{array} \begin{vmatrix} -\lambda & 4 & 0 \\ 4 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$$-\lambda \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} - 4 \begin{vmatrix} 4 & 1 \\ 0 & -\lambda \end{vmatrix} + 0 \begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix} = 0$$

$$+\lambda(\lambda^2 - 1) + 4(-4\lambda) = 0.$$

$$\lambda(\lambda^2 - 1) - 16 \cdot \lambda = 0.$$

$$\lambda[\lambda^2 - 1 - 16) = 0$$

$$\lambda[\lambda^2 - 17) = 0$$

$$\lambda_0 = 0 \qquad \lambda_1 = +\sqrt{17} \cdot a. \quad \lambda_2 = -\sqrt{17}\ a.$$

Val propres de $\hat{A} = \{0, +\sqrt{17}a, -\sqrt{17}a\}.$

* Vecteurs propres:

(i) $A|\vartheta_0\rangle = 0 \cdot |\vartheta_0\rangle. \qquad |\vartheta_0\rangle ??$

$$|\vartheta_0\rangle = a|u_1\rangle + b|u_2\rangle + c|u_3\rangle. = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} b = 0 \\ 4a + c = 0 \\ b = 0 \end{cases} \Rightarrow \begin{cases} b = 0 \\ \underline{\underline{c}} = -4a. \end{cases} \qquad a = -\frac{c}{4}$$

$|\vartheta_0\rangle$ normalisé: $\langle\vartheta_0|\vartheta_0\rangle = 1 = |a|^2 + |b|^2 + |c|^2 = \left|\frac{c}{4}\right|^2 + |c|^2$

$$1 = |a|^2 + 0^2 + 16 \cdot |a|^2 =$$

$$17|a|^2 = 1 \Rightarrow |a|^2 = \frac{1}{17} \Rightarrow |a| = \frac{1}{\sqrt{17}} \Rightarrow a = \frac{e^{i\theta}}{\sqrt{17}}. \quad (\theta = 0)$$

$$\boxed{a = \frac{1}{\sqrt{17}}}$$

$$|z| = a \Rightarrow \boxed{z = a \cdot e^{i\alpha}}$$

$$|\alpha| = \frac{1}{\sqrt{17}}$$

$$|z| = a \Rightarrow z = a \cdot e^{i\theta}$$

$$|\vartheta_3\rangle = \frac{1}{\sqrt{17}} \cdot |u_1\rangle - \frac{4}{\sqrt{17}} |u_3\rangle$$

- $\lambda_1 = \sqrt{17} \cdot a \rightsquigarrow |\vartheta_1\rangle$:

$$|\vartheta_1\rangle = a|u_1\rangle + b|u_2\rangle + c|u_3\rangle$$

$$\langle \vartheta_1|\vartheta_1\rangle = 1 = |a|^2 + |b|^2 + |c|^2.$$

$z \cdot \overrightarrow{\text{Vect}}$. propre: $A \cdot |\vartheta_1\rangle = \lambda_1 |\vartheta_1\rangle$

$$\cancel{a} \begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \sqrt{17} \cdot \cancel{a} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

$$\begin{cases} 4\beta = \sqrt{17} \cdot \alpha \\ 4\alpha + \gamma = \sqrt{17} \cdot \beta \\ \beta = \sqrt{17} \cdot \gamma \end{cases} \Rightarrow \begin{cases} \beta = \frac{\sqrt{17}}{4} \alpha \\ \\ \beta = \sqrt{17}\, \gamma \end{cases}$$

$$|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$$

$$\begin{matrix} 17 \\ 16 \\ 4 \end{matrix}$$

$$|\alpha|^2 + \frac{17}{16} \cdot |\alpha|^2 + \frac{1}{16} |\alpha|^2 = 1 \Rightarrow |\alpha|^2 \left(\frac{16}{16} + \frac{17}{16} + \frac{1}{16} \right) = 1 \Rightarrow |\alpha|^2 = \frac{16}{34}.$$

$$|\alpha|^2 \left(\frac{34}{16} \right) = 1 \Rightarrow |\alpha| = \frac{4}{\sqrt{34}}$$

$$\gamma = \sqrt{17}\, \underset{\sim}{\beta} - 4\alpha = \sqrt{17} \cdot \frac{\sqrt{17}}{4} \alpha - 4\alpha.$$

$$= \left(\frac{17}{4} - 4 \right) \alpha$$

$$\| \gamma = \frac{1}{4} \alpha. \|$$

$$|\vartheta_1\rangle = \frac{4}{\sqrt{34}} |u_1\rangle + \frac{\sqrt{17}}{4} \cdot \frac{4}{\sqrt{34}} |u_2\rangle + \frac{1}{\cancel{4}} \cdot \frac{\cancel{4}}{\sqrt{34}} \cdot |u_3\rangle$$

$$|\vartheta_1\rangle = \frac{1}{\sqrt{34}} \cdot \left(4|u_1\rangle + \sqrt{17} \cdot |u_2\rangle + |u_3\rangle. \right)$$

- $\overrightarrow{\text{Vect}}$. propre. $\lambda_2 = -\sqrt{17}\, a. \rightsquigarrow |\vartheta_2\rangle$:

$$A \cdot |\vartheta_2\rangle = \underset{\sim}{\lambda_2} |\vartheta_2\rangle \qquad |\vartheta_2\rangle = \alpha|u_1\rangle + \beta|u_2\rangle + \gamma|u_3\rangle$$

$$a\begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} = -\sqrt{17}\, a\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}$$

$$\begin{cases} 4\beta = -\sqrt{17}\,\alpha \\ 4\alpha + \gamma = -\sqrt{17}\,\beta \\ \beta = -\sqrt{17}\,\gamma \end{cases} \Rightarrow \begin{cases} \beta = -\dfrac{\sqrt{17}}{4}\cdot\alpha \\ +\dfrac{\cancel{\sqrt{17}}}{4}\,\alpha = +\cancel{\sqrt{17}}\,\gamma \\ \gamma = \dfrac{1}{4}\cdot\alpha \end{cases}$$

$$|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$$

$$|\alpha|^2 + \frac{17}{16}\cdot|\alpha|^2 + \frac{1}{16}|\alpha|^2 = 1$$

$$|\alpha|^2\cdot\left(\frac{16}{16} + \frac{17}{16} + \frac{1}{16}\right) = 1 \Rightarrow |\alpha|^2 = \frac{16}{34} \Rightarrow |\alpha| = \frac{4}{\sqrt{34}}$$

$$|v_2\rangle = \frac{4}{\sqrt{34}}\cdot|u_1\rangle - \frac{\sqrt{17}}{\cancel{4}}\cdot\frac{\cancel{4}}{\sqrt{34}}\cdot|u_2\rangle + \frac{1}{\cancel{4}}\,\frac{\cancel{4}}{\sqrt{34}}\,|u_3\rangle$$

$$\boxed{|v_2\rangle = \frac{1}{\sqrt{34}}\cdot\left(4|u_1\rangle - |u_2\rangle + |u_3\rangle\right)}$$

- $$\langle v_0|v_1\rangle = \frac{1}{\sqrt{34}}\,\frac{1}{\sqrt{17}}\cdot\left(\langle u_1| - 4\langle u_3|\right)\cdot\left(4.|u_1\rangle + \sqrt{17}.|u_2\rangle + |u_3\rangle\right)$$

$$= \frac{1}{\sqrt{34}}\cdot\frac{1}{\sqrt{17}}\cdot\Big(4.\underbrace{\langle u_1|u_1\rangle}_{1} + \sqrt{17}\,\underbrace{\langle u_1|u_2\rangle}_{0} + \underbrace{\langle u_1|u_3\rangle}_{0}$$

$$- 4.4\underbrace{\langle u_3|u_1\rangle}_{0} - 4.\sqrt{17}.\underbrace{\langle u_3|u_2\rangle}_{0} - 4.\underbrace{\langle u_3|u_3\rangle}_{1}\Big)$$

$$= \frac{1}{\sqrt{34}}\cdot\frac{1}{\sqrt{17}}\cdot(4-4) = 0.$$

$$\langle v_0|v_1\rangle = 0 \Rightarrow |v_0\rangle \perp |v_1\rangle$$

2)

$$H = \hbar\omega \begin{array}{c} \begin{array}{ccc} |u_1\rangle & |u_2\rangle & |u_3\rangle \end{array} \\ \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \end{array}$$

On considère le sous-espace $\{|u_1\rangle\,|u_2\rangle\}$

la matrice $\bar{H} = \hbar\omega \begin{array}{c} \begin{array}{cc} |u_1\rangle & |u_2\rangle \end{array} \\ \begin{pmatrix} 1 & -1 \end{pmatrix} \end{array}$

la matrice $\bar{H} = \hbar\omega \begin{array}{cc} |v_1\rangle & |v_2\rangle \end{array}$
$$\bar{H} = \hbar\omega \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\det(\bar{H} - \lambda . \mathbb{1}_{42}) = 0 \Rightarrow \begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(1-\lambda) - 1 = 0$$

$$\cancel{1} - 2\lambda + \lambda^2 - \cancel{1} = 0$$

$$\lambda^2 - 2\lambda = 0$$

$$\lambda(\lambda - 2) = 0$$

$$\begin{cases} \lambda = 0 \, \hbar\omega = 0 \\ \lambda = 2 . \hbar\omega \end{cases}$$

val propre de A $= \{0, 2\hbar\omega, -\hbar\omega\}$

(2.2)
$$\begin{cases} \bar{H} . |\varphi_0\rangle = 0 . |\varphi_0\rangle \\ \bar{H} . |\varphi_1\rangle = 2\hbar\omega \, |\varphi_1\rangle \\ H |v_3\rangle = -\hbar\omega \, |v_3\rangle \end{cases}$$

$$|\varphi_0\rangle = \alpha |v_1\rangle + \beta |v_2\rangle$$

$$|\varphi_0\rangle = \alpha |v_1\rangle + \beta |v_2\rangle + 0 |v_3\rangle .$$

$$\begin{cases} |\varphi_0\rangle = \frac{1}{\sqrt{2}} (|v_1\rangle + |v_2\rangle) \\ |\varphi_1\rangle = \frac{1}{\sqrt{2}} (|v_1\rangle - |v_2\rangle). \end{cases}$$

$\underline{\underline{H}} = H^{\dagger} \Rightarrow$ vect . orthogonaux .

$$\langle \varphi_0 | \varphi_1 \rangle = \langle \varphi_0 | v_3 \rangle = \langle \varphi_1 | v_3 \rangle = 0.$$

$\{A, H\}$ ECOC : $[A, H] = 0$

$$A \cdot H = \begin{pmatrix} \cdot & \cdot \\ \cdot & \end{pmatrix}$$

$$\neq$$

$$H \cdot A = \begin{pmatrix} \cdot & \\ & \end{pmatrix}$$

$$H \longrightarrow \boxed{|\psi\rangle} \longrightarrow E_0, E_1, E_3$$

$$P(E_0) = |\langle \varphi_0 | \psi_0 \rangle|^2$$

$$P(E_1) = |\langle \varphi_1 | \psi_0 \rangle|^2$$

$$P(E_3) = |\langle u_3 | \psi_0 \rangle|^2$$

après la mesure

$$H \longrightarrow \cancel{\boxed{|\psi_0\rangle}} \longrightarrow (-\hbar\omega) \longrightarrow \boxed{|u_3\rangle}$$

$$\begin{cases} P(E_0) = |\langle \varphi_0 | u_3 \rangle|^2 = 0 \\ P(E_1) = |\langle \varphi_1 | u_3 \rangle|^2 = 0 \\ P(E_2) = |\langle u_3 | u_3 \rangle|^2 = 1 \end{cases}$$

$$A \longrightarrow \{\lambda_0, \lambda_1, \lambda_2\}$$

$$\begin{cases} P(\lambda_0) = |\langle v_0 | u_3 \rangle|^2 = \frac{16}{17} \\ P(\lambda_1) = |\langle v_1 | u_3 \rangle|^2 = \frac{1}{34} \\ P(\lambda_2) = |\langle v_2 | u_3 \rangle|^2 = 1/34 \end{cases}$$

$$\| \sum P = 1 \|$$

c) $$\langle A \rangle_{|\psi(0)\rangle} = \langle \psi(0) | A | \psi(0) \rangle = \langle u_3 | A | u_3 \rangle$$

r) $\langle A \rangle_{|\psi(0)\rangle} = \langle \psi(0) | A | \psi(0) \rangle = \langle v_3 | A | v_3 \rangle$

$$= \sum_i \lambda_i \cdot P(\lambda_i).$$

$$\langle H \rangle_{|\psi(0)\rangle} = \langle \psi(0) | H | \psi(0) \rangle = \underbrace{\sum_{i=0}^{2} E_i \cdot P(E_i)}$$

r) $\underline{\underline{|\psi(t)\rangle}}$:

$$|\psi(t)\rangle = U(t,0)\,|\psi(t=0)\rangle$$

$$= \underbrace{\exp\left(-i\frac{\hat{H}t}{\hbar}\right)}\,|v_3\rangle$$

$$H.|v_3\rangle = -\hbar\omega\,|v_3\rangle$$

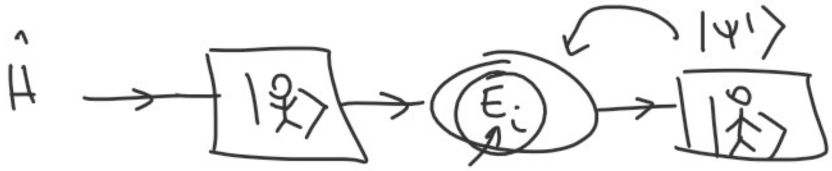
$$e^{-i\frac{\hat{H}t}{\hbar}}|v_3\rangle = e^{-i\left(-\frac{\cancel{\hbar}\omega}{\cancel{\hbar}}\right)t}|v_3\rangle$$

$$\| \; |\psi(t)\rangle = e^{i\omega t}\cdot|v_3\rangle \; \|$$

- valeurs propres = {resultat de mesure}

  Vect propres. = etat du système.

  operateurs (observable) = appareil de mesure.



$$P(E_i) = |\langle v_i | \quad \rangle|^2$$

$$|\psi'\rangle = \frac{P_i \,|\quad\rangle}{\sqrt{P_i\,|\quad\rangle}} \qquad \frac{v}{\|v\|}$$

$$\| \; P_i = |v_i\rangle\langle v_i| \; \|$$