Centre Be in sciences

Séances

Analyse II

ENSA-FST

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Séance 2 : Exercices les sommes de Riemann

Sommes :

- $\sum_{k=m}^{n} a = a \sum_{k=m}^{n} 1 = a.(n-m+1)$.

- $\sum_{k=m}^{n} k = \frac{(m-n+1)(m+n)}{2}$ $\quad \sum_{k=0}^{n} k = \frac{n(n+1)}{2}$ ✓

- $\sum_{0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ $\quad \cdot \sum_{0}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$

Exercice 24.2 *Déterminer* $\lim_{n\to+\infty} \sum_{k=n}^{2n-1} \frac{1}{n+k}$. $\quad \frac{1}{n} \cdot \sum f(x_k)$

- changement d'indice : $k' = k - n$ $\quad \begin{cases} k = n \to k' = 0 \\ k = 2n-1 \to k' = n-1 \end{cases}$

$$\sum_{k=n}^{2n-1} \frac{1}{n+k} = \sum_{k'=0}^{n-1} \frac{1}{n+k'+n} = \sum_{k'=0}^{n-1} \frac{1}{2n+k'} = \frac{1}{n} \cdot \sum_{0}^{n-1} \frac{1}{2+\frac{k'}{n}}$$

$$= \frac{1}{n} \cdot \sum_{0}^{n-1} f(x_k) \xrightarrow[n\to+\infty]{} \int_0^1 f(x)\,dx = \int_0^1 \frac{1}{2+x}\,dx = \left(\ln(2+x)\right]_0^1$$

$$\to \ln(3) - \ln(2) = \ln(3/2)$$

Exercice 24.3 *Calculer la limite de* $u_n = \frac{1}{n^2}\prod_{k=1}^{n}(n^2+k^2)^{\frac{1}{n}}$.

1. on a : $\ln(u_n) = \ln\left(\frac{1}{n^2}\cdot\prod_{1}^{n}(n^2+k^2)^{1/n}\right)$

$$= -\ln(n^2) + \sum \frac{1}{n}\ln(n^2+k^2)$$

$$= -2\ln(n) + \frac{1}{n}\cdot\sum_{1}^{n}\ln\left(n^2\left(1+\left(\frac{k}{n}\right)^2\right)\right)$$

$$= -2\ln(n) + \frac{1}{n}\cdot\sum_{1}^{n}\left(2\ln(n) + \ln\left(1+\left(\frac{k}{n}\right)^2\right)\right)$$

$$= -2\ln(n) + \frac{1}{n}\cdot\sum_{k=1}^{n} 2\ln(n) + \frac{1}{n}\sum_{1}^{n}\ln\left(1+\left(\frac{k}{n}\right)^2\right)$$

$$= \cancel{-2\ln(n)} + \frac{1}{n}\cdot 2\ln(n)\cancel{n} + \frac{1}{n}\cdot\sum_{1}^{n}\ln\left(1+\left(\frac{k}{n}\right)^2\right)$$

$$= \frac{1}{n}\sum_{1}^{n}\ln\left(1+\left(\frac{k}{n}\right)^2\right) \xrightarrow[n\to+\infty]{} \int_0^1 \ln(1+x^2)\,dx$$

Exercice 24.1 *Soit* $(u_n)_{n\in\mathbb{N}^*}$ *la suite définie par*

$$u_n = \sum_{k=1}^{n}\frac{n}{n^2+k^2}$$

Déterminer sa limite.

- $\sum_{k=1}^{n}\frac{n}{n^2\left(1+\left(\frac{k}{n}\right)^2\right)} = \frac{1}{n}\cdot\sum_{k=1}^{n}\frac{1}{1+\left(\frac{k}{n}\right)^2}$ $\quad x_k = \frac{k}{n}$.

$$= \frac{1}{n}\cdot\sum_{1}^{n}\frac{1}{1+x_k^2} \xrightarrow[n\to+\infty]{} \int_0^1 \frac{1}{1+x^2}\,dx$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \arctan(x)\Big|_0^1 = \frac{\pi}{4}.$$

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$$\int_0^1 \underbrace{1}_{v'}\cdot\underbrace{\ln(1+x^2)}_{u}\,dx = [\underbrace{\ln(1+x^2)}_{u}\cdot \underbrace{x}_{v}]_0^1 - 2\int_0^1 \underbrace{\frac{x^2+1-1}{x^2+1}}_{v\cdot u'}\,dx$$

$$= \ln(2) - 2\int_0^1 \left(1 - \frac{1}{x^2+1}\right)dx$$

$$= \ln(2) - 2\left[x - \operatorname{Arctg}(x)\right]_0^1$$

$$= \ln(2) - 2\left[\left(1-\frac{\pi}{4}\right) - (0 - \underbrace{\operatorname{Arctg}(0)}_{0})\right]$$

$$\ln(u_n) \longrightarrow \quad = \ln(2) + \frac{\pi}{2} - 2$$

donc $u_n \to \exp\left(\ln(2) - 2 + \frac{\pi}{2}\right)$

Exercice 24.4 *Déterminer la limite de* $u_n = \sqrt[n]{\frac{(2n)!}{n!n^n}}$.

$$\ln(u_n) = \ln\left(\sqrt[n]{\frac{(2n)!}{n!\,n^n}}\right) = \frac{1}{n}\ln\left(\frac{(2n)!}{n!\,n^n}\right)$$

$$= \frac{1}{n}\left\{\ln\left(\frac{(2n)!}{n!}\right) - n\ln(n)\right\}.$$

- $$\frac{(2n)!}{n!} = \frac{2n(2n-1)\cdots(2n-n)\cancel{n}\cdots\cancel{2\times1}}{\cancel{n}\times(\cancel{n-1})(\cancel{n-2})\cdots\cancel{\times2\times1}} = 2n(2n-1)\cdots(2n-n+1)$$

$$\xrightarrow{-1} \qquad \xleftarrow{+1}$$

$$= 2n(2n-1)\cdots(n+1)$$

$$= (n+1)(n+2)\cdots\times 2n = \prod_{1}^{n}(n+k)$$

$$\ln(2n!) = \sum_{k=1}^{2n}\ln(k)$$

$$\ln(n!) = \sum_{1}^{n}\ln k$$

$$\frac{(2n)!}{n!} = \prod_{1}^{n} (n+k) \qquad \| n.\ln(n) = \sum_{k=1}^{n} \ln(n) \|$$

$$\cdot \ln\left(\frac{(2n)!}{n!}\right) = \sum_{1}^{n} \ln(n+k).$$

$$\cdot \ln(v_n) = \frac{1}{n}\left\{ \sum_{1}^{n} \ln(n+k) - \sum_{k=1}^{n} \ln(n) \right\}$$

$$= \frac{1}{n}\left\{ \sum_{1}^{n} \ln\left(\frac{n+k}{n}\right)\right\} = \frac{1}{n}\cdot \sum_{1}^{n} \ln\left(1+\frac{k}{n}\right) \xrightarrow[+\infty]{} \int_0^1 \ln(1+x)\,dx$$

$$\int \ln(1+x)\,dx = \Big[(1+x)\ln(1+x) - (1+x)\Big]_0^1 \qquad x\ln$$

$$= 2\ln(2) - 2 + 1 = 2\ln(2) - 1$$

$$v_n \to e^{2\ln(2)-1} = e^{\ln 4} \times e^{-1} = 4e^{-1} = \frac{4}{e}$$

Exercice 24.5 *Déterminer la limite de* $u_n = \frac{1}{n}\sum_{k=0}^{n-1} \frac{k}{\sqrt{4n^2-k^2}}$.

$$\left(x_n = \frac{k}{n}\right)$$

$$u_n = \frac{1}{n}\cdot \sum_{0}^{n-1} \frac{k}{\sqrt{n^2\left(4-\left(\frac{k}{n}\right)^2\right)}} \qquad \frac{t'}{2\sqrt{t}}$$

$$= \frac{1}{n}\cdot \sum \frac{k}{n}\cdot \frac{1}{\sqrt{4-\left(\frac{k}{n}\right)^2}} \xrightarrow[+\infty]{} -\int_0^1 \frac{(4-x^2)'}{2\sqrt{4-x^2}}\,dx$$

$$= -\left[\sqrt{4-x^2}\right]_0^1 = -\sqrt{3} + \sqrt{4} = -\sqrt{3}+2$$

$$= 2 - \sqrt{3}$$

Exercice 24.6 *Déterminer la limite de* $u_n = \sum_{k=1}^{2n} \frac{k}{n^2+k^2}$.

$$\frac{b-a}{n}\sum f\left(\frac{b-a}{n}k\right) \to \int_a^b f(x)\,dx$$

$$U_n = \frac{2}{2n}\sum_{k=1}^{2n} \frac{\frac{k}{n}}{1+\left(\frac{k}{n}\right)^2}$$ n'est pas somme de Riemann.

$p = 2n$

$n = \frac{p}{2}$

$$U_p = \frac{2}{p}\sum_{k=1}^{p} \frac{\frac{k}{p}(2-0)}{1+\left(\frac{k}{p}(2-0)\right)^2} = \frac{b-a}{p}\sum_{k=0}^{p} f\left(\frac{k}{p}(b-a)\right)$$

$b = 2$

$a = 0$

$$\xrightarrow[p\to+\infty]{} \frac{1}{2}\int_0^2 \frac{(x^2+1)'}{1+x^2}\,dx.$$

Exercice 24.7 *Soit* $x \in \mathbb{R}\setminus\{-1,1\}$, *on pose* $f(x) = \int_0^{2\pi} \ln\left(x^2 - 2x\cos t + 1\right)dt$. $\quad t \in (0, 2\pi)$

1. *Déterminer* Df.
2. *Factoriser sur* $\mathbb{C}$ *le polynôme* $X^n - 1$.
3. *Calculer* $f(x)$ *à l'aide de ses sommes de Riemann.*

$$x^2 - 2x\cos t + 1 = x^2 - 2x\cos t + \cos^2 t + 1 - \cos^2 t$$
$$= \underbrace{(x-\cos t)^2} + \underbrace{\sin^2 t} \geqslant 0$$

$$(x-\cos t)^2 + \sin^2 t = 0 \iff \begin{cases} (x-\cos t)^2 = 0 \\ \sin^2 t = 0 \end{cases} \iff \begin{cases} x = \cos t \\ t = 0, \pi \end{cases}$$

$$\iff \begin{cases} x = \pm 1 \;\checkmark \\ t = 0\,(\pi) \end{cases}$$

$D_f = \mathbb{R} \setminus \{-1, 1\}$

2. $\underline{x^n - 1^n} = \prod_{0}^{n-1} \left(x - e^{2i\frac{k\pi}{n}}\right)$ ✓

$\frac{b-a}{n} \cdot \sum f\left(\frac{b-a}{n} \cdot k\right) \longrightarrow \int_a^b f(t)\,dt$

3. $\underline{S_n} \longrightarrow \int_0^{2\pi} f(t)\,dt$

$$S_n = \frac{2\pi - 0}{n} \cdot \sum_0^{n-1} \ln\left(x^2 - 2x\cdot\cos\left(\frac{2k\pi}{n}\right) + 1\right)$$

$$= \frac{2\pi}{n} \cdot \ln\left(\prod_0^{n-1} \left(x^2 - 2x \cos\left(\frac{2k\pi}{n}\right) + 1\right)\right)$$

$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

$$= \frac{2\pi}{n} \ln\left(\prod_0^{n-1} \left(x^2 - x\cdot e^{i2k\pi/n} - x\cdot e^{-i\frac{2k\pi}{n}} + 1\right)\right)$$

$$= \frac{2\pi}{n} \cdot \ln\left(\prod_0^{n-1} \left[\left(x - e^{i\frac{2k\pi}{n}}\right)\left(x - e^{-2i\frac{k\pi}{n}}\right)\right]\right)$$

$$= \frac{2\pi}{n} \cdot \ln\left(\prod\left(x - e^{2i\frac{k\pi}{n}}\right) \prod \cdot \left(x - e^{-2i\frac{k\pi}{n}}\right)\right)$$

$$= \frac{2\pi}{n} \cdot \ln\left(\left(x^n - 1\right)\left(x^n - 1\right)\right) = \frac{2\pi}{n} \cdot \ln\left(x^n - 1\right)^2 = \frac{4\pi}{n} \cdot \ln\left|x^n - 1\right|$$

- $$\prod\left(x - e^{-2i\frac{k\pi}{n}}\right) = \prod \overline{\left(x - e^{2i\frac{k\pi}{n}}\right)} = \overline{\prod\left(x - e^{2i\frac{k\pi}{n}}\right)}$$

$$= \overline{\left(x^n - 1\right)} = x^n - 1$$

$$\begin{cases} \left(\overline{z} = z \quad (z \in \mathbb{R})\right) \\ \prod\left(a_i b_i\right) = \prod a_i \prod b_i \end{cases}$$

$S_n = \frac{4\pi}{n} \cdot \ln|x^n - 1|$ $\qquad x^n - 1 \underset{+\infty}{\sim} x^n$

• si $|x| > 1$: $n \to +\infty$ $\quad S_n \underset{+\infty}{\sim} \frac{4\pi}{n} \cdot \ln x^n$

$\underset{+\infty}{\sim} 4\pi \cdot \ln|x| \cdot \underset{+\infty}{\longrightarrow} \boxed{4\pi \ln|x|}$

• $|x| < 1$: alors : $x^n \underset{+\infty}{\longrightarrow} 0$ $\quad \boxed{S_n \underset{+\infty}{\longrightarrow} 0.}$

donc : $\displaystyle\int_0^{2\pi} \ln(x^2 - 2x\cos t + 1)\,dt = 4\pi \ln|x|$. si $|x| > 1$

$= 0$ si $|x| < 1$.

Calculer $\int_0^1 x^2\,dx$ et $\int_0^1 x^3\,dx$ en utilisant des sommes de Riemman.

$f : [a,b] \to \mathbb{R}$ continue par morceaux.

$$\int_a^b f(x)\,dx = \lim_{n\to+\infty} \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + (b-a)\frac{k}{n}\right).$$

• $\displaystyle\int_0^1 x^2\,dx = \lim_{n\to+\infty} \frac{1-0}{n} \sum_1^n \left(\frac{k}{n}\right)^2 = \lim \frac{1}{n^3} \sum_1^n k^2$

$\displaystyle = \lim_{+\infty} \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \lim_{+\infty} \frac{(n+1)(2n+1)}{6n^2} \xrightarrow{+\infty} \frac{1}{3}$

Calculer $\lim_{n\to+\infty} S_n$, avec $S_n = \frac{1}{n\sqrt{n}} \sum_{k=1}^{n} \sqrt{k}$.

$$S_n = \frac{1}{n} \cdot \sum_{1}^{n} \frac{\sqrt{k}}{\sqrt{n}} = \frac{1-0}{n} \cdot \sum_{1}^{n} \underbrace{\left(\frac{k}{n}\right)^{1/2}}_{x_n} \longrightarrow \int_0^1 \sqrt{x}\, dx.$$

Soient f et g deux applications continues sur $[0,1]$.

Montrer que $\lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} f\left(\frac{k}{n}\right) g\left(\frac{k+1}{n}\right) = \int_0^1 f(x)g(x)\,dx$.

$$T_n = \frac{1}{n} \cdot \sum f\left(\frac{k}{n}\right) \cdot g\left(\frac{k}{n}\right) \xrightarrow[+\infty]{} \int_0^1 f(x)\, g(x)\, dx. \checkmark$$

$$\lim_{+\infty} (S_n - T_n) \longrightarrow 0$$

$$S_n - T_n = \frac{1}{n} \cdot \sum_{0}^{n-1} \left(f\left(\frac{k}{n}\right) \cdot \left(g\left(\frac{k+1}{n}\right) - g\left(\frac{k}{n}\right)\right)\right) \checkmark$$

g est continue sur $[0,1]$ $\Rightarrow$ uniformément continue. (Théo. Heine).

$\alpha > 0$ tq: $\forall (x,y) \in [0,1] \times [0,1]$: $|x-y| \leqslant \alpha \Rightarrow |g(x) - g(y)| \leqslant \varepsilon$.

$$\left|g\left(\frac{k+1}{n}\right) - g\left(\frac{k}{n}\right)\right| \leqslant \varepsilon.$$

Soient f et g deux applications continues sur $[0,1]$.

Montrer que $\lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} f\left(\frac{k}{n}\right)g\left(\frac{k+1}{n}\right) = \int_0^1 f(x)g(x)\,\mathrm{d}x$.

$$T_n = \frac{1}{n}\cdot \sum f\left(\frac{k}{n}\right)\cdot g\left(\frac{k}{n}\right) \xrightarrow[+\infty]{} \int_0^1 f(x)\,g(x)\,dx.$$

$$\lim_{+\infty} (S_n - T_n) \to 0$$

$$S_n - T_n = \frac{1}{n}\cdot \sum_{0}^{n-1} \left(f\left(\frac{k}{n}\right)\left(g\left(\frac{k+1}{n}\right) - g\left(\frac{k}{n}\right)\right)\right)$$

g est continue sur $[0,1]$ $\Rightarrow$ uniformément continue (Théo. Heine).

$\alpha > 0$ tq. $\forall (x,y) \in [0,1]\times[0,1]$: $|x-y| \leq \alpha \Rightarrow |g(x) - g(y)| \leq \varepsilon$.

$$\left|g\left(\frac{k+1}{n}\right) - g\left(\frac{k}{n}\right)\right| \leq \varepsilon.$$

$$\left|g\left(\frac{k+1}{n}\right) - g\left(\frac{k}{n}\right)\right| \leq \varepsilon.$$

$$\left|f\left(\frac{k}{n}\right)\right| \left| g\left(\frac{k+1}{n}\right) - g\left(\frac{k}{n}\right)\right| \leq \varepsilon \cdot \left|f\left(\frac{k}{n}\right)\right|$$

$$\left| f\left(\frac{k}{n}\right)\left(g\left(\frac{k+1}{n}\right) - g\left(\frac{k}{n}\right)\right)\right| \leq \varepsilon \cdot \left|f\left(\frac{k}{n}\right)\right|$$

$$|S_n - T_n| \leq \varepsilon \cdot U_n. \qquad U_n = \frac{1}{n}\sum_{0}^{n-1}\left|f\left(\frac{k}{n}\right)\right|$$

$$U_n = \frac{1}{n}\cdot \sum \left|f\left(\frac{k}{n}\right)\right| \to \int_0^1 |f(x)|\,dx.$$

(toute suite CV est bornée)

$$|S_n - T_n| \leq \varepsilon . M \leq \varepsilon. \quad (M \geq 0)$$

$$\to \boxed{\lim_{+\infty} (S_n - T_n) = 0} \qquad |u_n - v_n| < \varepsilon.$$

$$\lim S_n = \lim T_n = \int_0^1 f(x)\,dx$$